

Asymptotically conical Ricci flat Kahler metrics on \mathbb{C}^2 with cone singularities along a complex curve

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1 Introduction

We work on \mathbb{C}^2 with standard complex coordinates z, w . Let $P = P(z, w)$ be a degree d (≥ 2) polynomial such that $C = \{P = 0\}$ is a smooth complex curve. In this paper we restrict to the case in which C has d different asymptotic lines, i.e. the zero locus of the homogeneous degree d part of P consist of d distinct lines L_1, \dots, L_d .

We fix a number β such that

$$\frac{d-2}{d} < \beta < 1 \quad (1.1)$$

It is well known that under this condition 1.1 there exist a (unique up to scaling) compatible metric g on \mathbb{CP}^1 with constant positive gaussian curvature and cone angle β at the points corresponding to the lines L_1, \dots, L_d . If we normalize g to have curvature 1 what we mean is that around each singular point we can find polar coordinates (ρ, θ) in which $g = d\rho^2 + \beta^2 \sin^2(\rho) d\theta^2$ and g is locally isometric to the round sphere of radius 1 otherwise. (See Troyanov[14], Luo-Tian[10]).

There is a standard construction, known as 'the Calabi Ansatz' (see e.g. LeBrun[9] page 11), which produces a Ricci Flat Kahler cone metric out of a Kahler-Einstein metric with positive scalar curvature. In Section 2 we adapt this construction to the metric g to get a flat Kahler metric g_F on $\mathbb{C}^2 \setminus L$, where $L = \cup_{k=1}^d L_k$. This metric is singular along L . More precisely, around each point $0 \neq p \in L$ we can find holomorphic coordinates (z_1, z_2) centered at p in which $g_F = |z_1|^{2\beta-2} |dz_1|^2 + |dz_2|^2$. The property of g_F that we shall exploit the most is the one of being a metric cone, with its apex at 0.

We denote by \mathcal{D} the set of all diffeomorphisms F of \mathbb{C}^2 for which there exist a compact K such that $F(C \setminus K) \subset L$ and are asymptotic to the identity in the following sense: there exist constants A_j such that $|F(x) - x| \leq A_0$, $|DF(x) - Id| \leq A_1 |x|^{-1}$ and $|D^\alpha F(x)| \leq A_j |x|^{-j}$ for all $x \in \mathbb{C}^2$ and $j = |\alpha| \geq 2$. It is elementary to prove that \mathcal{D} is not empty, see 3.1.

Finally let us denote by r the intrinsic distance in g_F to 0 and let $\Omega = (1/\sqrt{2}) dz \wedge dw$. We are now ready to state our main result.

Theorem 1 *There exist a Kahler metric g_{RF} on $\mathbb{C}^2 \setminus C$ and $H \in \mathcal{D}$ such that*

- $$\omega_{RF}^2 = |P|^{2\beta-2} \Omega \wedge \overline{\Omega} \quad (1.2)$$

where ω_{RF} is the associated Kahler form, and

- $$|(H^{-1})^* g_{RF} - g_F|_{g_F} \leq A r^\gamma \quad (1.3)$$

outside a compact set, for some constants $A > 0$ and $\gamma < 0$.

It follows from the proof of Theorem 1 that g_{RF} has cone singularities in the sense of Donaldson[5], this notion will be recalled in Subsection 4.1. But let us say for now that g_{RF} is smooth on $\mathbb{C}^2 \setminus C$ and that around each point $p \in C$ we can find holomorphic coordinates (z_1, z_2) centered at p and a number $\Lambda > 0$ such that $\Lambda^{-1} g_{(\beta)} \leq g_{RF} \leq \Lambda g_{(\beta)}$ where $g_{(\beta)} = |z_1|^{2\beta-2} |dz_1|^2 + |dz_2|^2$. In the case that $d = 2$ we can assume that $C = \{zw = 1\}$, the metrics of Theorem 1 have S^1 symmetry and were constructed in Section 5 of [5] by means of the Gibbons-Hawking ansatz.

Let us now provide some context for Theorem 1 following the lines of Section 5 and 6 of [5]. Let X be a closed complex surface and $C_\epsilon \subset X$ be smooth complex curves for $\epsilon > 0$. Let $0 < \beta < 1$ be fixed and assume that we have Kahler metrics ω_ϵ with cone angle $2\pi\beta$ along C_ϵ and $\text{Ric}(\omega_\epsilon) = \omega_\epsilon$, say, on the complement of the curve. Let $p \in X$ and suppose that the curves C_ϵ converge to a curve C_0 singular at p . If the singularity is modeled on $\{P_d = 0\}$ (with P_d a homogeneous degree $d \geq 2$ polynomial) we would expect that (under favorable conditions) after re-scaling ω_ϵ around small balls centered at p we will get a metric g_{RF} of the kind given by our Theorem 1. For a more specific example see Section 6.

We will now outline the strategy we follow to prove Theorem 1.

In Section 3 we construct $H \in \mathcal{D}$ and a reference metric ω_{ref} which has cone angle β along C and is asymptotic to ω_F in the sense of 1.3. In Subsection 3.3 we prove, following Appendix A in Jeffres-Mazzeo-Rubinstein [7], that ω_{ref} has bisectional curvature bounded by above. Later on this will be crucial.

We start Section 4 by reviewing some material from [5]. Here we say what we mean by a metric with a cone singularity and state the interior Schauder estimates (Proposition 3), which are of fundamental importance in our analysis. Having the interior estimates at hand, in subsections 4.2 and 4.3 we develop a theory of 'weighted Holder spaces'. Our main references in doing this are Pacard[11] and Bartnik[1], see also Chapter 8 in [13]. The main result of Section 4 is Proposition 5. This parallels known results in the case of asymptotically conical smooth metrics as stated in Conlon-Hein[4] (Theorem 2.11).

In subsection 4.4, as an application of Proposition 5, we show the existence of a metric ω_0 asymptotic to ω_F such that

$$\omega_0^2 = e^{-f} |P|^{2\beta-2} \Omega \wedge \bar{\Omega} \quad (1.4)$$

with f a smooth function of compact support. What will be important for us, apart from the fast decay of f , is that 1.4 implies that ω_0 has bounded Ricci curvature (in fact the bound $\text{Ric}(\omega_0) \geq -B\omega_0$ for some $B > 0$ is the bound that will be relevant to us).

To prove the theorem it is enough to show that there exist $u \in C_\delta^{2,\alpha}$ (our notation for the weighted Holder spaces) such that

$$(\omega_0 + i\partial\bar{\partial}u)^2 = e^f \omega_0^2$$

for then $\omega_{RF} = \omega_0 + i\partial\bar{\partial}u$ will be our solution (the positivity of ω_{RF} follows from the equation, the decay of $\partial\bar{\partial}u$ and the connectedness of $\mathbb{C}^2 \setminus C$). In order to do this we use the continuity method and consider the set

$$T = \{t \in [0, 1] : \exists u_t \in C_\delta^{2,\alpha} \text{ solving } (\omega_0 + i\partial\bar{\partial}u_t)^2 = e^{tf} \omega_0^2\} \quad (1.5)$$

We want to prove that $1 \in T$. Proposition 5 implies that T is open and $0 \in T$ trivially ($u_0 = 0$). The closedness of T follows from the a priori estimate $\|u_t\|_{2,\alpha,\delta} \leq C$ for some constant $C > 0$. This is the content of Proposition 7 (the main point being that C is independent of $t \in T$). We prove this proposition into several steps. First we estimate the C^0 norm of u , to do this we use the Sobolev inequality (for the metric ω_0) and then we run a Moser iteration following Chapter 8 of Joyce[8]. To estimate the C^2 norm of u we use the maximum principle and the Chern-Lu inequality (in a slightly different way than in [7]). Here it is crucial that we have an upper bound on the bisectional curvature of ω_{ref} and a lower bound on the Ricci curvature of $\omega_t = \omega_0 + i\partial\bar{\partial}u_t$ in the form of $\text{Ric}(\omega_t) \geq -A\omega_{ref}$. For some $A > 0$. This bound holds for ω_0 by 1.4 and it holds for ω_t since along the continuity path 1.5

$$\text{Ric}(\omega_t) = (1 - t)\text{Ric}(\omega_0) \quad (1.6)$$

The C^2 estimate gives us the uniform bound $C^{-1}\omega \leq \omega_t \leq C\omega$. Then we can apply the interior $C^{2,\alpha}$ estimate given by Theorem 1.7 of Chen-Wang [3].

Finally we proceed to the weighted estimates. We start by proving a bound on $\|u_t\|_{0,\mu}$ for some $\delta < \mu < 0$. The technique is again Moser iteration, we follow [8]. Finally the bound on $\|u_t\|_{2,\alpha,\delta}$ follows from the linear theory developed.

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2 Flat metrics.

We use the notation from above. Let $L_k = \{l_k = 0\}$ with l_k linear functions of z, w for $k = 1, \dots, d$. So that $L = \cup_{k=1}^d L_k = \{P_d = l_1 \dots l_d = 0\}$ where P_d is the homogeneous degree d part of P . The main result of this section is the following

Proposition 1 *There exists a Kahler metric g_F on $\mathbb{C}^2 \setminus L$ with Kahler form ω_F such that*

$$\omega_F^2 = |P_d|^{2\beta-2} \Omega \wedge \overline{\Omega} \quad (2.1)$$

The metric is a cone with apex at 0, is invariant under the S^1 action $e^{it}(z, w) = (e^{it}z, e^{it}w)$ and

$$\omega_F = \frac{i}{2} \partial \bar{\partial} r^2 \quad (2.2)$$

where r is the intrinsic distance to 0.

As we said this result is a consequence of the fact that under the condition 1.1 there exist a compatible metric g on \mathbb{CP}^1 with constant positive gaussian curvature and cone angle β at the points corresponding to the lines L . It will turn out at the end that g is a Kahler quotient of g_F by the S^1 action.

Let us point that $(d-2)/d < \beta$ is a necessary condition for the existence of such a metric g . In fact, Gauss-Bonnet tells us that

$$2 + d\beta - d = \frac{1}{2\pi} \int_{\mathbb{CP}^1} K_g dV_g \quad (2.3)$$

Finally we point out some other properties of the metrics given by Proposition 1 that follow from the proof of it

- For every $p \notin L$ we can find holomorphic coordinates (z_1, z_2) around p such that the metric is given by $|dz_1|^2 + |dz_2|^2$. Because of this we refer to these metrics as 'Flat metrics'.
- For every $0 \neq p \in L$ we can find holomorphic coordinates (z_1, z_2) on a neighborhood U around p such that $U \cap L = \{z_1 = 0\}$ and the metric is given by $|z_1|^{2\beta-2} |dz_1|^2 + |dz_2|^2$.
- Let $\lambda > 0$ and denote $m_\lambda(z, w) = (\lambda z, \lambda w)$. The neighborhoods in the previous two items can be taken to be invariant under m_λ and $r^2 \circ m_\lambda = \lambda^c r^2$ for all $\lambda > 0$, where $c = 2 + d\beta - d$. Note that 1.1 means that $0 < c < 2$.

2.1 Hopf bundle and singular metrics on the 3-sphere

In this section we construct metrics on the 3-sphere with cone singularities of angle $2\pi\beta$ along the Hopf circles corresponding to L . Let us start by describing a local model for the singularities.

Denote by $g_{(\beta)}$ the singular metric on \mathbb{C}^2 given by $g_{(\beta)} = |z_1|^{2\beta-2} |dz_1|^2 + |dz_2|^2$. We want to write $g_{(\beta)}$ as a metric cone, in order to do that we first set $z_1 = (\beta r_1)^{1/\beta} e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ so that $g_{(\beta)} = dr_1^2 + \beta^2 r_1^2 d\theta_1^2 + dr_2^2 + r_2^2 d\theta_2^2$. Finally define $r \in (0, \infty)$ and $\rho \in (0, \pi/2)$ by $r_1 = r \sin \rho$, $r_2 = r \cos \rho$ to get $g_{(\beta)} = dr^2 + r^2 \overline{g}_{(\beta)}$, where

$$\overline{g}_{(\beta)} = d\rho^2 + \beta^2 \sin^2(\rho) d\theta_1^2 + \cos^2(\rho) d\theta_2^2 \quad (2.4)$$

. We think of $\overline{g}_{(\beta)}$ as a metric on the 3-sphere with a cone singularity of angle $2\pi\beta$ transverse to the circle given by the intersection of $\{z_1 = 0\}$ with the unit sphere in \mathbb{C}^2 . We refer to this space as S_β^3 .

Let $S^3 = \{|z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$ and $H : S^3 \rightarrow \mathbb{CP}^1$ be the Hopf bundle. Denote by g the compatible metric on \mathbb{CP}^1 with constant curvature $K_g = 4$ and cone angle β at the points corresponding to L .

Lemma 1 *There exist an S^1 invariant metric \overline{g} on $S^3 \setminus L$ such that*

- $H : (S^3 \setminus L, \overline{g}) \rightarrow (\mathbb{CP}^1 \setminus L, g)$ is a riemannian submersion.
- \overline{g} is locally isometric to the round 3-sphere of radius 1

- each $p \in L$ has a neighborhood which maps isometrically to a neighborhood of a singular point in S^3_β

Proof:

We begin by writing g in coordinates. W.l.o.g. we can assume that $L_j = \{z = a_j w\}$ with $a_j \in \mathbb{C}$ for $j = 1, \dots, d-1$ and $L_d = \{w = 0\}$. Write $\xi = z/w$, then $g = e^{2\phi}|d\xi|^2$ with ϕ a function of ξ .

Consider the function

$$u = \phi - (\beta - 1) \sum_{j=1}^{d-1} \log |\xi - a_j|$$

The point of defining u in this way is that around each a_j there exist holomorphic coordinates with $\eta(a_j) = 0$ in which

$$g = \beta^2 \frac{|\eta|^{2\beta-2}}{(1 + |\eta|^{2\beta})^2} |d\eta|^2$$

, so that $\phi = \log \beta + (\beta - 1) \log |\eta| - \log(1 + |\eta|^{2\beta})$. It is easy to check from here that u is a continuous function on \mathbb{C} and that

$$\lim_{\xi \rightarrow a_j} |\xi - a_j| \frac{\partial u}{\partial \xi} = 0$$

for $j = 1, \dots, d-1$.

On $\mathbb{C} \setminus \{a_1, \dots, a_{d-1}\}$ define the real 1-form

$$\alpha_0 = \frac{i}{c} (\partial u - \bar{\partial} u) \quad (2.5)$$

where $c = 2 + d\beta - d$. Then it follows that

$$\lim_{\epsilon \rightarrow 0} \int_{C_\epsilon(a_j)} \alpha_0 = 0 \quad (2.6)$$

where $C_\epsilon(a_j) = \{|\xi - a_j| = \epsilon\}$ for $j = 1, \dots, d-1$.

On the other hand

$$d\alpha_0 = -\frac{2i}{c} \partial \bar{\partial} u = \frac{1}{c} K_g dV_g \quad (2.7)$$

so that 2.3 gives

$$\frac{1}{2\pi} \int_{\mathbb{C}} d\alpha_0 = 1 \quad (2.8)$$

On the trivial S^1 -bundle $\mathbb{C} \setminus \{a_1, \dots, a_{d-1}\} \times S^1$ with coordinates (ξ, e^{it}) consider the connection $\alpha = dt + \alpha_0$ and the metric

$$\bar{g} = g + \frac{c^2}{4} \alpha^2 \quad (2.9)$$

Let $p = (\xi_0, e^{it_0}) \in \mathbb{C} \setminus \{a_1, \dots, a_{d-1}\} \times S^1$, we want to prove that \bar{g} is isometric to $S^3(1)$ in a neighborhood of p . We can find polar coordinates (ρ, θ) around ξ_0 in which

$$g = d\rho^2 + \frac{\sin^2(2\rho)}{4} d\theta^2$$

In this coordinates $d\alpha_0 = (1/c) K_g dV_g = (2/c) \sin(2\rho) d\rho d\theta$. Doing a change of gauge if necessary we can assume $\alpha_0 = (2/c) \sin^2(\rho) d\theta$. In this coordinates $(c/2)\alpha = (c/2)dt + \sin^2(\rho) d\theta$. If we assume $t_0 \in (-\pi, \pi)$, say, and define $\bar{t} = (c/2)t$ we finally get

$$\bar{g} = d\rho^2 + \frac{\sin^2(2\rho)}{4} d\theta^2 + (d\bar{t} + \sin^2(\rho) d\theta)^2$$

which can be recognized as $S^3(1)$.

We use the map

$$(\xi, e^{it}) \rightarrow \left(z = \frac{\xi}{\sqrt{1+|\xi|^2}} e^{it}, w = \frac{1}{\sqrt{1+|\xi|^2}} e^{it} \right)$$

to think of \bar{g} as a metric on $S^3 \setminus L$. The S^1 invariance and the first item of the lemma are clear from the definition 2.9 of \bar{g} . We already checked the second item so let's prove the last one.

Assume first that $p \in L_j$ for some $1 \leq j \leq d-1$. Write $p = (a_j, e^{it_0})$. We can find polar coordinates (ρ, θ) around a_j in which

$$g = d\rho^2 + \beta^2 \frac{\sin^2(2\rho)}{4} d\theta^2$$

In this coordinates $d\alpha_0 = (1/c)K_g dV_g = (2/c)\beta \sin(2\rho) d\rho d\theta$. It follows from 2.6 that we can perform a change of gauge so that $\alpha_0 = (2/c)\beta \sin^2(\rho) d\theta$. In this coordinates $(c/2)\alpha = (c/2)dt + \beta \sin^2(\rho) d\theta$. If we assume $t_0 \in (-\pi, \pi)$, say, and define $\bar{t} = (c/2)t$ we have that

$$\bar{g} = d\rho^2 + \beta^2 \frac{\sin^2(2\rho)}{4} d\theta^2 + (d\bar{t} + \beta \sin^2(\rho) d\theta)^2$$

Write $\theta_2 = \bar{t}$, $\theta_1 = \theta + \beta^{-1}\bar{t}$ to get $\bar{g} = d\rho^2 + \beta^2 \sin^2(\rho) d\theta_1^2 + \cos^2(\rho) d\theta_2^2$, which matches with the expression 2.4 of the metric in S_β^3 .

Finally consider the case of $p \in L_d = \{w = 0\}$. In the coordinates

$$(\eta, e^{is}) \rightarrow \left(z = \frac{1}{\sqrt{1+|\eta|^2}} e^{is}, w = \frac{\eta}{\sqrt{1+|\eta|^2}} e^{is} \right)$$

we have $p = (0, e^{is_0})$. These coordinates are related to (ξ, e^{it}) via $\eta = 1/\xi$ and $e^{is} = (\xi/|\xi|)e^{it}$. So that $\alpha = dt + \alpha_0 = ds + \beta_0$ with $\beta_0 = d(\arg \eta) + \alpha_0$. Now $\lim_{\epsilon \rightarrow 0} \int_{|\eta|=\epsilon} \alpha_0 = -\lim_{N \rightarrow \infty} \int_{|\xi|=N} \alpha_0$. It follows from 2.6, 2.8 and Stokes' theorem that $\lim_{N \rightarrow \infty} \int_{|\xi|=N} \alpha_0 = 2\pi$. As a result $\lim_{\epsilon \rightarrow 0} \int_{|\eta|=\epsilon} \beta_0 = 0$. From here we can proceed as before, finding polar coordinates in which $g = d\rho^2 + \beta^2 \frac{\sin^2(2\rho)}{4} d\theta^2$ and changing gauge so that $\beta_0 = (2/c)\beta \sin^2(\rho) d\theta$.

□

Remark 1 *The proof above gives us that the fibers of H have constant length πc . Since $\text{Vol}(g) = (\pi/2)c$ we have $\text{Vol}(\bar{g}) = (\pi^2/2)c^2$.*

In a coordinate free way we can say that the metric of Lemma 1 is given by 2.9. Where α is the connection on the Hopf bundle with $d\alpha = c^{-1}H^*(K_g dV_g)$ which satisfies the following gauge fixing condition:

- If $p \in \mathbb{CP}^1$ is a point in L and γ_ϵ is a loop that shrinks to p as $\epsilon \rightarrow 0$, then the holonomy of α along γ_ϵ goes to the identity as $\epsilon \rightarrow 0$.

2.2 Proof of Proposition 1

On $\mathbb{R}_{>0} \times \mathbb{C} \setminus \{a_1, \dots, a_{d-1}\} \times S^1$ with coordinates (r, ξ, e^{it}) define

$$g_F = dr^2 + r^2 \bar{g} \tag{2.10}$$

Write $\xi = x + iy$. Consider the almost-complex structure given by

$$I \frac{\tilde{\partial}}{\partial x} = \frac{\tilde{\partial}}{\partial y}, \quad I \frac{\partial}{\partial r} = \frac{2}{cr} \frac{\partial}{\partial t}$$

where

$$\frac{\tilde{\partial}}{\partial x} = \frac{\partial}{\partial x} - \alpha \left(\frac{\partial}{\partial x} \right) \frac{\partial}{\partial t}, \quad \frac{\tilde{\partial}}{\partial y} = \frac{\partial}{\partial y} - \alpha \left(\frac{\partial}{\partial y} \right) \frac{\partial}{\partial t}$$

are the horizontal lifts of $\partial/\partial x$ and $\partial/\partial y$. Finally set $\omega_F = g_F(I, \cdot)$.

Claim 1 $(\mathbb{R}_{>0} \times \mathbb{C} \setminus \{a_1, \dots, a_{d-1}\} \times S^1, g_F, I)$ is a Kahler manifold. I.e. $d\omega_F = 0$ and I is integrable. Moreover,

$$\omega_F = \frac{i}{2} \partial \bar{\partial} r^2 \quad (2.11)$$

Proof:

We compute in the coframe $\{dx, dy, dr, \alpha\}$ where

$$\omega_F = r^2 e^{2\phi} dx \wedge dy + \frac{cr}{2} dr \wedge \alpha$$

so that $d\omega_F = 2re^{2\phi} dr dx dy - (cr/2)(4/c)e^{2\phi} dr dx dy = 0$. The integrability of I amounts to check that

$$\left[\frac{\tilde{\partial}}{\partial x} + i \frac{\tilde{\partial}}{\partial y}, \frac{\partial}{\partial r} + i \frac{2}{cr} \frac{\partial}{\partial t} \right] = 0$$

Finally $dId(r^2) = d(2rIdr) = -cd(r^2\alpha) = -2crdr \wedge \alpha - 4r^2 e^{2\phi} dx \wedge dy$. Using that $2i\partial\bar{\partial} = -dId$ we deduce 2.11 □

Claim 2 The functions $w = (c/2)^{1/c} r^{2/c} e^{u/c} e^{it}$ and $z = \xi w$ give a biholomorphism between $\mathbb{R}_{>0} \times \mathbb{C} \setminus \{a_1, \dots, a_{d-1}\} \times S^1$ with the complex structure I and $\mathbb{C}^2 \setminus L$. Under this map

$$\omega_F^2 = |P_d|^{2\beta-2} \Omega \wedge \bar{\Omega} \quad (2.12)$$

Proof:

It is easy to see that the pair (z, w) defines a diffeomorphism between the corresponding spaces. The Cauchy-Riemann equations for a function h to be holomorphic with respect to I are given by

$$\frac{\partial h}{\partial r} + i \frac{2}{cr} \frac{\partial h}{\partial t} = 0, \quad \frac{\partial h}{\partial x} + i \frac{\partial h}{\partial y} = \alpha \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \frac{\partial h}{\partial t}$$

if we ask h to have weight 1 with respect to the circle action then

$$\frac{\partial h}{\partial r} = \frac{2}{cr} h \quad \frac{\partial h}{\partial \xi} = i\alpha \left(\frac{\partial}{\partial \xi} \right) h = \frac{1}{c} \frac{\partial u}{\partial \xi} h$$

From here we see that z and w are holomorphic.

One can check that $\omega_F^2 = cr^3 e^{2\phi} dx dy dr dt$ and $\Omega \wedge \bar{\Omega} = (4/c)|w|^4 r^{-1} dx dy dr dt$, so that

$$\omega_F^2 = (c^2/4)|w|^{-4} r^4 e^{2\phi} \Omega \wedge \bar{\Omega}$$

Now we use that $r^4 = (4/c^2)|w|^{2c} e^{-2u}$, $\phi - u = (\beta - 1) \sum_{j=1}^{d-1} \log |(z/w) - a_j|$ and $2c - 4 = 2d\beta - 2d = 2\beta - 2 + (d-1)(2\beta - 2)$ to conclude that

$$\omega_F^2 = |z - a_1 w|^{2\beta-2} \dots |z - a_{d-1} w|^{2\beta-2} |w|^{2\beta-2} \Omega \wedge \bar{\Omega}$$

which is 2.12. □

From the formula for the function w in the previous claim we get

$$r^2 = \frac{2}{c} |w|^c e^{-u} \quad (2.13)$$

We recall that

$$u = \phi - (\beta - 1) \sum_{j=1}^{d-1} \log |\xi - a_j| \quad , \quad g = e^{2\phi} |d\xi|^2 \quad (2.14)$$

. Where ϕ a function of $\xi = z/w$. We are writing the lines as $L_j = \{z = a_j w\}$ with $a_j \in \mathbb{C}$ for $j = 1, \dots, d-1$ and $L_d = \{w = 0\}$.

2.13 together with 2.14 give a recipe to go from the metric g_F in \mathbb{C}^2 in Proposition 1 to the corresponding g on \mathbb{CP}^1 and vice versa.

Finally we introduce some notation that will be useful later. We denote

$$\tau_1 = e^\phi r d\xi, \quad \tau_2 = dr + i \frac{cr}{2} \alpha \quad (2.15)$$

. Then $\{\tau_1, \tau_2\}$ is (up to a factor of $\sqrt{2}$) an orthonormal basis for the $(1,0)$ forms in $\mathbb{C}^2 \setminus L$. I.e $\omega_F = (i/2)\tau_1 \bar{\tau}_1 + (i/2)\tau_2 \bar{\tau}_2$. Let us denote points in the 3-sphere by θ and for $\lambda > 0$ define $D_\lambda(r, \theta) = (\lambda r, \theta)$. In complex coordinates we have $D_\lambda = m_{\lambda^{2/c}}$, i.e $D_\lambda(z, w) = (\lambda^{2/c} z, \lambda^{2/c} w)$. From 2.15 we have $D_\lambda^* \tau_1 = \lambda \tau_1$ and $D_\lambda^* \tau_2 = \lambda \tau_2$. The forms τ_1, τ_2 are not holomorphic.

2.3 Examples and remarks

We begin by giving some examples with which we test the equations 2.13 2.14.

When $d = 2$, for any $0 < \beta < 1$ we know that

$$g = \beta^2 \frac{|\xi|^{2\beta}}{(1 + |\xi|^{2\beta})^2} |d\xi|^2$$

from 2.13 we get $r^2 = \beta^{-2}(|z|^{2\beta} + |w|^{2\beta})$ so that $g_F = |z|^{2\beta-2}|dz|^2 + |w|^{2\beta-2}|dw|^2$. We refer to this space as $\mathbb{C}_\beta \times \mathbb{C}_\beta$.

Next we claim that when $d = 3$ and $\beta = 1/2$ the metric g_F is a quotient of the euclidean metric. We take our lines to be $L_1 = \{z = 0\}$, $L_2 = \{z = w\}$ and $L_3 = \{w = 0\}$. Let $D_4 \subset SU(2)$ be the subgroup generated by $(x, y) \rightarrow (ix, -iy)$ and $(x, y) \rightarrow (-y, x)$. The polynomials $z = (x^2 + y^2)^2$, $w = (x^2 - y^2)^2$ and $t = 2(x^5 y - y^5 x)$ are invariant under the action and give the complex isomorphism

$$\mathbb{C}^2/D_4 \cong \{zw(z-w) = t^2\}$$

. Let $G \subset U(2)$ be the subgroup generated by D_4 and $(x, y) \rightarrow (y, x)$. Then $D_4 \subset G$ is normal and $K = G/D_4 \cong \mathbb{Z}_2$ acts on \mathbb{C}^2/D_4 as $(z, w, t) \rightarrow (z, w, -t)$. The functions z, w give an isomorphism of complex manifolds $\mathbb{C}^2/G \cong \mathbb{C}^2$. We can push forward the euclidean metric $\omega_{euc} = (i/2)\partial\bar{\partial}(|x|^2 + |y|^2)$ to obtain a flat Kahler metric with cone angle $\beta = 1/2$ along L .

From the formulas for z, w we have that $|z| + |w| = 2|x|^4 + 2|y|^4$ and $|z - w| = 4|x|^2|y|^2$ so that $2(|x|^2 + |y|^2)^2 = |z| + |w| + |z - w|$. From here we get that

$$r^2 = a(|z| + |w| + |z - w|)^{1/2} \quad (2.16)$$

where $a = 8\sqrt{2}$ is determined by the normalization condition 2.1. We can now use the equations 2.13 2.14 to get

$$g = \frac{1}{8} \frac{1}{|\xi||\xi-1| + |\xi|^2|\xi-1| + |\xi||\xi-1|^2} |d\xi|^2$$

Indeed, the map $\Phi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ given by $\Phi(x, y) = ((x^2 + y^2)^2, (x^2 - y^2)^2)$ maps lines to lines and induces $F : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ given by

$$\xi = F(\eta) = \frac{(\eta^2 + 1)^2}{(\eta^2 - 1)^2} \quad (2.17)$$

. Then one can check that $F^*g = (1 + |\eta|^2)^{-2}|d\eta|^2$ (the smooth metric with constant curvature 4). The map F has degree 4 and has six critical points at $0, \pm 1, \pm i, \infty$ and it maps the spherical triangle $T = \{|\eta| \leq 1, 0 \leq \arg(\eta) \leq \pi/2\}$ to the upper half plane $H = \{\text{Im}(\xi) \geq 0\}$. Then we recognize g as the metric obtained by gluing two copies of T along the boundary.

The fact is that when $d = 3$ and $1/3 < \beta < 1$ the metric g is given by gluing two copies of the spherical equilateral triangle T with interior angles equal to $\beta\pi$. If G is a conformal equivalence between H and T then g will be the pull back by G of the smooth constant curvature metric on T , extended to

\mathbb{C} by requiring the conjugation map to be an isometry. The construction of such a map G is related to the study of the hypergeometric equation. See Chapter 15 in [6].

Finally let us mention a different approach to Proposition 1, corresponding to the one in page 11 of LeBrun [9]. We think of \mathbb{C}^2 as the total space of $\mathcal{O}_{\mathbb{CP}^1}(-1)$ with the zero section collapsed at 0. The bundle projection is given by $\Pi : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{CP}^1$, $\Pi(z, w) = [z : w]$. We can then identify (smooth) hermitian metrics on $\mathcal{O}_{\mathbb{CP}^1}(-1)$ with (smooth) functions $h : \mathbb{C}^2 \rightarrow \mathbb{R}_{\geq 0}$ such that $h(\lambda p) = |\lambda|^2 h(p)$ for all $\lambda \in \mathbb{C}$, $p \in \mathbb{C}^2$ and $h(p) = 0$ only when $p = 0$.

The first basic fact we need is that an area form ω in \mathbb{CP}^1 induces an hermitian metric h_ω . We use coordinates $\xi = z/w$, $\eta = w/z$ on \mathbb{CP}^1 . Write $\omega = e^{2\phi}(i/2)d\xi d\bar{\xi}$ with $\phi = \phi(\xi)$ on $U = \Pi(\{w \neq 0\})$ and $\omega = e^{2\psi}(i/2)d\eta d\bar{\eta}$ with $\psi = \psi(\eta)$ on $V = \Pi(\{z \neq 0\})$. Then h_ω is given by

$$h_\omega = |w|^2 e^{-\phi}, \text{ if } w \neq 0; \quad h_\omega = |z|^2 e^{-\psi}, \text{ if } z \neq 0 \quad (2.18)$$

The second basic fact is that an hermitian metric h gives a 2-form ω_h on \mathbb{CP}^1 by means of

$$\omega_h = i\partial\bar{\partial} \log h(\xi, 1) \text{ on } U, \text{ and } \omega_h = i\partial\bar{\partial} \log h(1, \eta) \text{ on } V \quad (2.19)$$

We also mention that h induces hermitian metrics on the other complex line bundles over \mathbb{CP}^1 . If we regard P_d as a section of $\mathcal{O}_{\mathbb{CP}^1}(d)$, then we have $|P_d|_h^2 = h(\xi, 1)^{-d} |\xi - a_1|^2 \dots |\xi - a_{d-1}|^2$ on U and a corresponding expression on V .

One can then rephrase then the existence of g by saying that there exist an hermitian metric h , continuous on \mathbb{C}^2 and smooth outside L such that

$$h = |P_d|_h^{\beta-1} h_{\omega_h} \quad (2.20)$$

Where by $|P_d|_h$ we mean $|P_d|_h \circ \Pi$. (In fact we can be more precise and instead of saying that h is continuous we can give a local model for h around points of L .)

From 2.20 one gets that ω_h has constant gaussian curvature equal to $c = 2 + d\beta - d$ outside L and one can also argue that $(2\pi)^{-1} \int_{\mathbb{CP}^1} \omega_h = 1$.

The potential for ω_F is then given by $r^2 = ah^{c/2}$ for some constant $a > 0$ determined by 2.1.

3 Reference Metrics

The main result of this section is the following

Proposition 2 *There exist $H \in \mathcal{D}$ and Kahler metrics ω, ω_{ref} on \mathbb{C}^2 with cone singularities of angle β along C such that*

- $|(H^{-1})^* \omega - \omega_F|_{g_F} = O(r^{-2/c})$
- $Bisec(\omega_{ref}) \leq B$
- $Q^{-1} \omega_{ref} \leq \omega \leq Q \omega_{ref}$

for some positive constants Q, B .

We give the definition of a metric having cone singularities in 4.1. The statement about the singularities will follow from the fact around points of C one can write the metrics as $(\text{smth}) + i\partial\bar{\partial}(F|z_1|^{2\beta})$ with (smth) a smooth $(1, 1)$ form and F a smooth positive function. (See Lemma 10).

The metric ω is isometric to the flat metric ω_F in a neighborhood of C at infinity. In the notation of the next subsection this neighborhood is $U_{\delta/2, 2R}$.

3.1 A diffeomorphism

As before, let $C = \{P = 0\}$, the homogeneous degree d part of P being given by $l = l_1 \dots l_d$. We write $l_j = z - a_j w$, for $j = 1, \dots, d-1$ and $l_d = w$. W.l.o.g. let us assume that $a_j \neq 0$ for all $j = 1, \dots, d-1$.

First look at the piece of C which is asymptotic to $L_d = \{w = 0\}$.

Lemma 2 *There exist $R, \delta > 0$ and $\Phi = \Phi(z) : \{|z| > R\} \rightarrow \mathbb{C}$ bounded holomorphic, which depend only on P such that*

$$C \cap U_{d,\delta,R} = \{(z, \Phi(z))\}$$

where $U_{d,\delta,R} = \{|w| < \delta|z|, |z| > R\}$.

Proof:

Let S_j for $j = 1, \dots, d-1$ be orthogonal linear transformations taking L_d to L_j , write $U_{j,\delta,R} = S_j(U_{d,\delta,R})$ and $U_{\delta,R} = \cup_{j=1}^d U_{j,\delta,R}$. Taking δ small enough we can assume that the sets $U_{j,\delta,R}$ are pairwise disjoint. Write

$$P = P_d + Q \tag{3.1}$$

with Q a polynomial of degree $d-1$. On the complement of $U_{\delta,R}$ we have that $|P_d(x)| \geq C_1|x|^d$ for some $C_1 > 0$. Since $\deg(Q) = d-1$ we can find $C_2 > 0$ such that $|Q(x)| \leq C_2|x|^{d-1}$. It follows that for R big enough

$$C \cap \{|z| > R\} \subset U_{\delta,R} \tag{3.2}$$

For each z with $|z| > R$ we write

$$P(z, w) = P_z(w) = a(w - h_1(z)) \dots (w - h_d(z)) \tag{3.3}$$

With $a = (-1)^{d-1}a_1 \dots a_{d-1} \neq 0$ and $h_j : \{|z| > R\} \rightarrow \mathbb{C}$ holomorphic.

It follows from 3.2 that for each j , $\{(z, h_j(z))\} \subset U_{i,\delta,R}$ for some $i = i(j)$. In particular this implies that there exist a constant $A > 0$ such that

$$|h_j(z)| \leq A|z| \tag{3.4}$$

for $j = 1, \dots, d$.

We want to show that we can label the functions h_j in a way such that $i(j) = j$. First we note that if $i(j_0) = d$ then h_{j_0} is bounded. Indeed $|l_1 \dots l_{d-1}(x)| \geq c|x|^{d-1}$ for some $c > 0$ and all $x \in U_{d,\delta,R}$, so that $|h_{j_0}(z)| = |Q|/|l_1 \dots l_{d-1}| \leq C_2/c$.

It follows from 3.3 that the coefficient in front of w in the polynomial $P_z(w)$ is given by

$$(-1)^{d-1}a \sum_{j=1}^d \prod_{i \neq j} h_i(z) \tag{3.5}$$

On the other hand 3.1 and $P_d = w(z - a_1 w) \dots (z - a_{d-1} w)$, imply that 3.5 is a polynomial of degree $d-1$ in z (with leading term z^{d-1}). If we had $i(j_0) = i(j_1) = d$ for some $j_0 \neq j_1$ then h_{j_0} and h_{j_1} would be bounded. This together with the bound 3.4 would imply that the absolute value of 3.5 would be bounded by a constant times $|z|^{d-2}$, contradicting 3.5 being a degree $d-1$ polynomial.

Changing coordinates we can argue the same way for the other asymptotic lines. We conclude that the map $j \rightarrow i(j)$ is injective and we can perform the desired labeling. The lemma follows by setting $\Phi = h_d$.

In fact $h_j(z) = (1/a_j)z + \phi_j(z)$ with ϕ_j bounded for $j = 1, \dots, d-1$ so that 3.3 gives

$$P(z, w) = (l_1 + \phi_1) \dots (l_{d-1} + \phi_{d-1})(w - \Phi) \tag{3.6}$$

□

Lemma 3 *Let $\delta > 0$ be small enough and $R > 0$ big enough, then there exists a diffeomorphism $H \in \mathcal{D}$ such that H is holomorphic in $U_{\delta/2,2R}$ and H is the identity outside $U_{\delta,R}$.*

Proof: Let $\chi = \chi(t)$ be a smooth cut-off function with $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) = 0$ for $t \geq 2$. We start by defining H in the region asymptotic to L_d . Let

$$h(z, w) = \chi\left(\frac{2|w|}{\delta|z|}\right) (1 - \chi)(R|z|) \quad (3.7)$$

It follows that $h = 1$ on $U_{d,\delta/2,2R}$, $h = 0$ outside $U_{d,\delta,R}$ and $|D^\alpha h(x)| \leq C_{|\alpha|}|x|^{-|\alpha|}$ for any multi-index α . We set

$$H_d(z, w) = (z, w - h\Phi) \quad (3.8)$$

Since Φ is a bounded holomorphic function of z and in the region $U_{d,\delta,R}$ we have $|z| \geq c|(z, w)|$ for some $c > 0$, we conclude that there exist constants A_j such that $|H_d(x) - x| \leq A_0$, $|DH_d(x) - Id| \leq A_1|x|^{-1}$ and $|D^\alpha H_d(x)| \leq A_j|x|^{-j}$ for all $x \in \mathbb{C}^2$ and $j = |\alpha| \geq 2$.

We proceed similarly for the other asymptotic regions, and in an obvious notation we set

$$H = H_1 \circ \dots \circ H_d \quad (3.9)$$

□

From now on we fix $\delta, R > 0$ and H .

3.2 Construction of ω

We start by deriving some consequences of

$$r^2 \circ m_\lambda = \lambda^c r^2 \quad (3.10)$$

for all $\lambda > 0$ and $c = 2 + d\beta - d$. First of all we get that $m_\lambda^* \omega_F = \lambda^c \omega_F$. Since ω_F is positive we can find $a > 0$ such that $\omega_F \geq a\omega_{euc}$ on the euclidean unit sphere, the scaling property then gives

$$\omega_F(p) \geq a|p|^{c-2}\omega_{euc} \quad (3.11)$$

For every $p \in \mathbb{C}^2$. ($|p|$ denotes the euclidean norm).

On the other hand, from the continuity of r one gets

$$b^{-1}|p|^c \leq r^2(p) \leq b|p|^c \quad (3.12)$$

for some $b > 0$.

Differentiating equation 3.10 on $\mathbb{C}^2 \setminus L$ we get that $D^\alpha r \circ m_\lambda = \lambda^{c-|\alpha|} D^\alpha r$ for any multi-index α . For $\epsilon > 0$ denote $U_\epsilon = U_{\epsilon,0}$, with the notation as in the previous subsection. From the smoothness of r on the complement of L it follows that

$$|D^\alpha r^2(p)| \leq A|p|^{c-|\alpha|} \quad (3.13)$$

on $\mathbb{C}^2 \setminus U_\epsilon$, where the constant A depends on ϵ and $|\alpha|$.

It follows from 3.13 and 3.11 that in the complement of U_ϵ there exist $a_\epsilon > 0$ such that

$$a_\epsilon|p|^{c-2}\omega_{euc} \leq \omega_F(p) \leq a_\epsilon^{-1}|p|^{c-2}\omega_{euc} \quad (3.14)$$

Let us denote by I the complex structure of \mathbb{C}^2 and let G be the inverse of H .

Lemma 4

$$|G^*I - I|_{g_F} = O(r^{-2/c}) \quad (3.15)$$

Proof:

First we note that $|G^*I - I|_{g_{euc}} = O(|p|^{-1})$ since it is given basically by $\bar{\partial}G$. From 3.12 we get $O(|p|^{-1}) = O(r^{-2/c})$. Secondly, there exist $\epsilon > 0$ such that G is holomorphic in $U_{2\epsilon}$. (More precisely this is true outside a compact set). So $G^*I = I$ in $U_{2\epsilon}$.

Note that in a vector space with an inner product the norm of an endomorphism doesn't change if we multiply the inner product by a positive constant. Hence $|G^*I - I|_{|p|^{c-2}g_{euc}} = O(|p|^{-1})$. Finally 3.14 gives the lemma.

□

We move on and define

$$\eta = \frac{i}{2} \partial \bar{\partial} (r^2 \circ H) \quad (3.16)$$

Lemma 5 *There exist a compact K such that $\eta > 0$ outside K . Moreover,*

$$|G^* \eta - \omega_F|_{g_F} = O(r^{-2/c})$$

Proof:

Denote $H(z, w) = (u, v)$, so that $r^2 = r^2(u, v)$. Write $U = U_{\delta, R}$ and $U' = U_{\delta/2, 2R}$, the subsets introduced in the previous subsection.

We remove compact sets whenever necessary. Note that $G^* \eta = \omega_F$ in $H(U')$, clearly we can pick $\epsilon > 0$ such that $U_\epsilon \subset H(U')$. In $\mathbb{C}^2 \setminus H(U')$ we are then able to use the bounds 3.13. Set $p_0 = (u_0, v_0) = H(x_0)$ with $x_0 = (z_0, w_0) \notin U'$ we start by computing $\eta(x_0)$

$$\begin{aligned} \frac{\partial}{\partial z} (r^2 \circ H) &= \frac{\partial r^2}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial r^2}{\partial \bar{u}} \frac{\partial \bar{u}}{\partial z} + \frac{\partial r^2}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial r^2}{\partial \bar{v}} \frac{\partial \bar{v}}{\partial z} \\ \frac{\partial^2}{\partial z \partial \bar{z}} (r^2 \circ H) &= \frac{\partial^2 r^2}{\partial^2 u} \frac{\partial u}{\partial \bar{z}} \frac{\partial u}{\partial z} + \frac{\partial^2 r^2}{\partial u \partial \bar{u}} \frac{\partial \bar{u}}{\partial \bar{z}} \frac{\partial u}{\partial z} + \frac{\partial^2 r^2}{\partial u \partial v} \frac{\partial v}{\partial \bar{z}} \frac{\partial u}{\partial z} + \frac{\partial^2 r^2}{\partial u \partial \bar{v}} \frac{\partial \bar{v}}{\partial \bar{z}} \frac{\partial u}{\partial z} + \frac{\partial r^2}{\partial u} \frac{\partial^2 u}{\partial z \partial \bar{z}} \\ &\quad + (\dots) \end{aligned}$$

where (\dots) consist of 15 terms that the reader can figure out.

Note that the second term is equal to

$$\frac{\partial^2 r^2}{\partial u \partial \bar{u}}(p_0) (1 + O(|x|^{-1}))$$

The first, third and fourth terms can be bounded by $A|x|^{c-2}|x|^{-1}$ and the fifth by $A|x|^{c-1}|x|^{-2}$ for some constant $A > 0$. It is easy to see that the remaining 15 terms can be bounded by $A|x|^{c-2}|x|^{-1}$ (the ones which contain second derivatives of r^2) or $A|x|^{c-1}|x|^{-2}$ (the ones which contain second derivatives of H). We conclude that we can bound all this terms by a constant times $|x|^{c-3}$.

We argue similarly for the other derivatives in $\partial \bar{\partial} (r^2 \circ H)$ to conclude that

$$G^* \eta(p_0) = \omega_F(p_0) + O(|x|^{c-3}) dz d\bar{z} + O(|x|^{c-3}) dz d\bar{w} + O(|x|^{c-3}) dw d\bar{z} + O(|x|^{c-3}) dw d\bar{w}$$

Note that $dz d\bar{z} = du d\bar{u} + \nu$ where ν is a 2-form with $|\nu|_{euc} = O(|p|^{-1})$. From 3.11 we get $|du d\bar{u}|_{g_F} = O(|x|^{2-c})$. We argue equally for the other terms to conclude that

$$|G^* \eta - \omega_F|_{g_F}(p_0) = O(|p_0|^{-1}) \quad (3.17)$$

3.12 then gives the result. □

Remark 2 *As we already said, $G^*(\eta) = \omega_F$ on a region $U_{\delta', R'}$ for some $\delta', R' > 0$. In the complement of this region one can extend 3.17 to*

$$|\nabla^i (G^* \eta - \omega_F)|_{g_F}(p_0) = O(r^{-(2/c)-i}) \quad (3.18)$$

where ∇ is the Levi-Civita connection of g_F .

We continue with the construction of ω . Let h be a cut-off function with $h = 1$ on B_N (the euclidean ball of radius N , say) and $h = 0$ on B_{N+1}^c where N is large enough so that $C \cap B_N^c \subset U'$ and $\eta > 0$ outside B_N . Consider

$$\omega' = \frac{i}{2} \partial \bar{\partial} (h|P|^{2\beta} + (1-h)(r^2 \circ H)) \quad (3.19)$$

Note that $\omega' = \eta > 0$ on B_{N+1}^c . On the other hand

$$\omega' = \frac{i}{2} \partial \bar{\partial} |P|^{2\beta} = \beta^2 |P|^{2\beta-2} \frac{i}{2} \partial P \wedge \bar{\partial} \bar{P} \geq 0$$

on B_N . Finally consider the annulus $B_{N+1} \setminus B_N$

Claim 3 *There exist $a > 0$ such that $\omega' \geq -a\omega_{euc}$ on $B_{N+1} \setminus B_N$.*

Proof: Indeed, for $x \in C \cap (B_{N+1} \setminus B_N)$ we can find holomorphic coordinates (z_1, z_2) such that $C = \{z_1 = 0\}$ and $r^2 \circ H = |z_1|^{2\beta} + |z_2|^2$. In this coordinates $P = fz_1$ for some non-vanishing holomorphic f . Then we have $2\omega' = i\partial\bar{\partial}(h(|f|^{2\beta}|z_1|^{2\beta}) + (1-h)(|z_1|^{2\beta} + |z_2|^2)) = (\text{smooth}) + i\partial\bar{\partial}u$ where $u = |z_1|^{2\beta}(h|f|^{2\beta} + 1 - h)$. On a smaller neighborhood we can assume $|f|^{2\beta} \geq \epsilon > 0$ so that $i\partial\bar{\partial}u = iu\partial\log u \wedge \bar{\partial}\log u + ui\partial\bar{\partial}\log u \geq ui\partial\bar{\partial}\log F$ where $F = h|f|^{2\beta} + 1 - h$. Note that F is smooth and $F \geq \min\{\epsilon, 1\}$ to conclude the claim. \square

Lemma 6 *There exist a Kahler metric on \mathbb{C}^2 with cone singularities of angle β along C such that $\omega = \eta$ outside a compact set.*

Proof: Let $\chi = \chi(t)$ be a smooth cut-off function with $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) = 0$ for $t \geq 2$. For $L > 0$ and $x \in \mathbb{C}^2$ let $\chi_L(x) = \chi(L^{-1}|x|)$. Set $\phi = \log(1 + |z|^2 + |w|^2)$ and define

$$\omega_L = \omega' + iK\partial\bar{\partial}(\chi_L\phi) \quad (3.20)$$

with $K > 0$ such that $Ki\partial\bar{\partial}\phi + \omega' > 0$ and $L > N + 2$. If L is big enough we can assume that on the annulus on $B_{2L} \setminus B_L$, $\omega' = \eta$. Recall that $|\eta|_{euc} \geq C_1|x|^{c-2}$ on the other hand, on $B_{2L} \setminus B_L$ we can bound $|\partial\bar{\partial}(\chi_L\phi)|_{euc} \leq C_2|x|^{-2}\log|x|$ (with C_2 independent of L). Taking L large we get that ω_L is positive everywhere. Fix such a large L and define $\omega = \omega_L$. The statement about the cone singularities follows from Lemma 10. \square

For reference in the future we say something about the volume form of ω . Define a function f in \mathbb{C}^2 by means of the equation

$$\omega^2 = e^f |P|^{2\beta-2} \Omega \wedge \bar{\Omega} \quad (3.21)$$

Lemma 7 *Outside a compact set f is a smooth function with*

$$|D^\alpha f(x)| \leq A_{|\alpha|} |x|^{-1-|\alpha|} \quad (3.22)$$

Proof:

Consider first the complement of $U_{\delta,R}$, where H is the identity and $\eta = \omega_F$. Compare 2.1 and 3.21 to obtain

$$e^f = |P|^{2-2\beta} |l|^{2\beta-2} = \left| 1 + \frac{Q}{l} \right|^{2-2\beta}$$

in the complement of $U_{\delta,R}$ we have constants $b_{|\alpha|}$ such that

$$|D^\alpha(Q/l)|(x) \leq b_{|\alpha|} |x|^{-1-|\alpha|}$$

3.22 then follows from $f = (2 - 2\beta) \log |1 + Q/l|$.

Secondly we consider the region $U_{\delta/2,2R}$, where H is holomorphic and $\eta = H^*\omega_F$. We see that $e^f = |P/(l \circ H)|^{2-2\beta}$. We focus in $U_{\delta/2,2R}$ and use 3.6 to get $P/(l \circ H) = (1 + \psi_1(z)) \dots (1 + \psi_{d-1}(z))$

where $\psi_j(z)$ are holomorphic with $|\psi_j(z)| \leq A|z|^{-1}$ for some $A > 0$. Note that in $U_{d,\delta/2,2R}$ we have $|z| \geq a|(z, w)|$ for some $a > 0$. As before we get 3.22.

Finally consider the region $U_{\delta,R} \setminus U_{\delta/2,2R}$. By Lemma 5 we can write $\eta = H^*\omega_F + \xi$ where ξ is a 2-form with $|\xi|_{g_F} = O(|x|^{-1})$. We conclude that $\eta^2 = (1 + O(|x|^{-1})) H^*\omega_F^2$ and we can proceed as before. \square

3.3 Upper bound on $\text{Bisec}(\omega_{ref})$

We start by defining ω_{ref} . Fix $0 < \delta < c$. Let h be a smooth function such that $h(p) = |p \circ H|^\delta$ outside a compact set K and $\nu = i\partial\bar{\partial}h \geq 0$ in all of \mathbb{C}^2 . Note that outside K there is $a > 0$ such that

$$a^{-1}|p|^{\delta-2}\omega_{euc} \leq \nu(p) \leq a|p|^{\delta-2}\omega_{euc}$$

We define ω_{ref} as

$$\omega_{ref} = \omega + \Lambda\nu \quad (3.23)$$

where $\Lambda > 0$ will be specified later on. From the definition it follows that

$$Q^{-1}\omega \leq \omega_{ref} \leq Q\omega \quad (3.24)$$

for some $Q > 0$.

The goal is to prove the following

Lemma 8

$$\text{Bisec}(\omega_{ref}) \leq B$$

Let us start by recalling the definition of bisectonal curvature. Let ω be a Kahler metric on an open subset U of \mathbb{C}^2 . For $x \in U$ and $v, w \in T_x^{1,0}\mathbb{C}^2$ with $|v|_\omega = |w|_\omega = 1$ we set

$$\text{Bisec}_\omega(v, w) = R(v, \bar{v}, w, \bar{w})$$

where R is the Riemann curvature tensor of ω . Recall that if (z_1, z_2) are holomorphic coordinates around x in which $\omega = \sum_{i,j=1}^2 g_{i\bar{j}} i dz_i d\bar{z}_j$ and $v = v_1 \partial/\partial z_1 + v_2 \partial/\partial z_2$, $w = w_1 \partial/\partial z_1 + w_2 \partial/\partial z_2$ then

$$\text{Bisec}_\omega(v, w) = \sum_{i,j,k,l=1}^2 R_{i\bar{j}k\bar{l}} v_i \bar{v}_j w_k \bar{w}_l$$

where

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j},k\bar{l}} + \sum_{s,t=1}^2 g^{s\bar{t}} g_{i\bar{t},k} g_{s\bar{j},\bar{l}}$$

(indexes after the comma indicate differentiation and $(g^{i\bar{j}})$ denotes the inverse transpose of the positive hermitian matrix $(g_{i\bar{j}})$, the index i being for the rows and j for the columns).

In Appendix A of [7] it is shown that if η is a smooth Kahler form in the unit ball $B_1 \subset \mathbb{C}^2$, say, and F is a smooth positive function such that

$$\omega = \eta + i\partial\bar{\partial}(F|z_1|^{2\beta}) \quad (3.25)$$

is Kahler on $B_1 \setminus \{z_1 = 0\}$. Then there exist a number C such that $\text{Bisec}(\omega) \leq C$ on $B_{1/2} \setminus \{z_1 = 0\}$, say. We choose $\Lambda > 0$ in 3.23 such that ω_{ref} can be written in the form 3.25 around the points of the curve. Then [7] gives us an upper bound on $\text{Bisec}(\omega_{ref})$ on compact sets. In order to extend this bound to \mathbb{C}^2 we use the 'asymptotically conical' behavior of ω_{ref} .

At points $x \in C$ where $\chi_L(x) = 0$ the metric ω in Lemma 6 can't be written in the form 3.25. (Compare with the one in Lemma 10). I haven't been able to get an upper bound on $\text{Bisec}(\omega)$ around such points.

To prove Lemma 8 it suffices to bound from above $\text{Bisec}(\omega_F + G^*\nu)$ in a region U_{δ_0, R_0} for some $\delta_0, R_0 > 0$. Note that outside a compact set $G^*\nu = i\partial\bar{\partial}|p|^\delta$.

Let $0 \neq q \in L$ and B a neighborhood of q where there exist coordinates (ξ_1, ξ_2) which map B to the unit ball in \mathbb{C}^2 in which $\omega_F = |\xi_1|^{2\beta-2}id\xi_1d\bar{\xi}_1 + id\xi_2d\bar{\xi}_2$. We might also assume that $|q| \geq 2$ and that B is contained in the euclidean ball of radius half the euclidean distance from q to 0.

Let $m_\lambda : B \rightarrow \lambda B$ for $\lambda \geq 1$ be the multiplication by λ in \mathbb{C}^2 . We simplify notation and write ν for $G^*\nu$. Then

$$m_\lambda^*(\omega_F + \nu) = \lambda^c(\omega_F + \lambda^{\delta-c}\nu)$$

We will show that we have an upper bound for the bisectonal curvature of $\omega_F + \lambda^{-c}m_\lambda^*\nu$ on $B_{1/2}$ which is independent of $\lambda \geq 1$. By a covering argument this gives the desired bound on U_{δ_0, R_0} and hence proves Lemma 8.

Write $\nu_{i\bar{j}} = \nu(\frac{\partial}{\partial\xi_i}, \frac{\partial}{\partial\bar{\xi}_j})$. Let $Q > 0$ be such that

$$Q^{-1}(\delta_{ij}) \leq (\nu_{i\bar{j}}) \leq Q(\delta_{ij}); \quad |\nu_{i\bar{j},k}| \leq C; \quad |\nu_{i\bar{j},k\bar{l}}| \leq C \quad (3.26)$$

on $B_{1/2}$.

Write $\omega = \omega_{(\beta)} + \epsilon\nu$ with ν a smooth Kahler form in the unit ball in \mathbb{C}^2 , $0 < \epsilon < 1$,

$$\omega_{(\beta)} = |\xi_1|^{2\beta-2}id\xi_1d\bar{\xi}_1 + id\xi_2d\bar{\xi}_2$$

and

$$\nu = \sum_{i,j=1}^2 \nu_{i\bar{j}}id\xi_id\bar{\xi}_j$$

The desired bound then follows from the following

Lemma 9 *There exist a constant C , independent of ϵ such that $\text{Bisec}(\omega) \leq C$ on $B_{1/2}$. In fact C depends only on Q , where $Q > 0$ is such that on $B_{1/2}$ $Q^{-1}\omega_{euc} \leq \nu \leq Q\omega_{euc}$ and $|\nu_{i\bar{j},k}|, |\nu_{i\bar{j},k\bar{l}}| \leq Q$ for any i, j, k, l .*

Proof: (This follows in the same lines as the one in Appendix A in [7])

Write

$$\omega = |\xi_1|^{2\beta-2}id\xi_1d\bar{\xi}_1 + \sum_{i,j=1}^2 \tilde{g}_{i\bar{j}}id\xi_id\bar{\xi}_j$$

So that,

$$\tilde{g}_{1\bar{1}} = \epsilon\nu_{1\bar{1}}, \quad \tilde{g}_{1\bar{2}} = \epsilon\nu_{1\bar{2}}, \quad \tilde{g}_{2\bar{2}} = 1 + \epsilon\nu_{2\bar{2}}$$

Let $x = (x_1, x_2) \in B_{1/2} \setminus \{\xi_1 = 0\}$. Define new coordinates (z_1, z_2) around x via

$$\begin{aligned} \xi_1 &= z_1 \\ \xi_2 &= z_2 + \frac{a}{2}(z_1 - x_1)^2 + b(z_1 - x_1)(z_2 - x_2) + \frac{c}{2}(z_2 - x_2)^2 \end{aligned}$$

where

$$a = -(\tilde{g}_{2\bar{2}}(x))^{-1}\tilde{g}_{1\bar{2},1}(x), \quad b = -(\tilde{g}_{2\bar{2}}(x))^{-1}\tilde{g}_{1\bar{2},2}(x), \quad c = -(\tilde{g}_{2\bar{2}}(x))^{-1}\tilde{g}_{2\bar{2},2}(x)$$

In this new coordinates we have

$$\omega = |z_1|^{2\beta-2}idz_1d\bar{z}_1 + \sum_{i,j} \hat{g}_{i\bar{j}}idz_id\bar{z}_j$$

Claim 4 $\hat{g}_{i\bar{j},k}(x) = 0$ when $j \neq 0$.

Indeed, write $d\xi_2 = Adz_1 + Bdz_2$, with $A = a(z_1 - x_1) + b(z_2 - x_2)$ and $B = 1 + b(z_1 - x_1) + c(z_2 - x_2)$. A straightforward computation gives

$$\hat{g}_{1\bar{2}} = \tilde{g}_{1\bar{2}}\bar{B} + \tilde{g}_{2\bar{2}}A\bar{B}, \quad \hat{g}_{2\bar{2}} = |B|^2\tilde{g}_{2\bar{2}}$$

from which we get

$$\hat{g}_{1\bar{2},1}(x) = \tilde{g}_{1\bar{2},1}(x) + \tilde{g}_{2\bar{2}}(x)a, \quad \hat{g}_{1\bar{2},2}(x) = \tilde{g}_{1\bar{2},2}(x) + \tilde{g}_{2\bar{2}}(x)b, \quad \hat{g}_{2\bar{2},2}(x) = \tilde{g}_{2\bar{2},2}(x) + \tilde{g}_{2\bar{2}}(x)c$$

Our choice of a, b, c implies that these three numbers are zero. The Kahler condition $\hat{g}_{i\bar{j},k} = \hat{g}_{k\bar{j},i}$ implies that $\hat{g}_{2\bar{2},1}(x) = \hat{g}_{1\bar{2},2}(x) = 0$ and the claim follows.

We compute the bisecional curvature of ω at x using the coordinates (z_1, z_2) . Let $v = v_1\partial/\partial z_1 + v_2\partial/\partial z_2$ and $w = w_1\partial/\partial z_1 + w_2\partial/\partial z_2 \in T_x^{1,0}\mathbb{C}^2$ with $|v|_\omega = |w|_\omega = 1$. Note that this implies that $|v_1|, |w_1| \leq C|z_1|^{1-\beta}$ and $|v_2|, |w_2| \leq C$. Write $\omega = \sum_{i,j=1}^2 g_{i\bar{j}}dz_i d\bar{z}_j$. So that $g_{i\bar{j}} = \hat{g}_{i\bar{j}}$ when $(i, j) \neq (1, 1)$ and $g_{1\bar{1}} = |z_1|^{2\beta-2} + \hat{g}_{1\bar{1}}$. Write $\text{Bisec}_\omega(v, w) = T_1 + T_2$, where

$$T_1 = - \sum_{i,j,k,l} g_{i\bar{j},k\bar{l}}(x) v_i \bar{v}_j w_k \bar{w}_l$$

and

$$T_2 = \sum_{s,t,i,j,k,l=1}^2 g^{s\bar{t}}(x) g_{i\bar{t},k}(x) g_{s\bar{j},\bar{l}}(x) v_i \bar{v}_j w_k \bar{w}_l$$

Claim 5

$$T_1 \leq C - (\beta - 1)^2 |z_1|^{2\beta-4} |v_1|^2 |w_1|^2$$

In fact $g_{1\bar{1},1\bar{1}} = (\beta - 1)^2 |z_1|^{2\beta-4} + \hat{g}_{1\bar{1},1\bar{1}}$, and we have

$$\hat{g}_{1\bar{1}} = \tilde{g}_{1\bar{1}} + A\tilde{g}_{2\bar{1}} + \bar{A}\tilde{g}_{1\bar{2}} + |A|^2\tilde{g}_{2\bar{2}}$$

from here we compute

$$\hat{g}_{1\bar{1},1\bar{1}}(x) = \tilde{g}_{1\bar{1},1\bar{1}}(x) + a\tilde{g}_{2\bar{1},1\bar{1}}(x) + \bar{a}\tilde{g}_{1\bar{2},1\bar{1}}(x) + |a|^2\tilde{g}_{2\bar{2}}$$

Since the differential at x of the change of coordinates between (ξ_1, ξ_2) and (z_1, z_2) is the identity, we have that

$$\tilde{g}_{i\bar{j},k}(x) = \frac{\partial \tilde{g}_{i\bar{j}}}{\partial \xi_k}(x) = \frac{\partial \nu_{i\bar{j}}}{\partial \xi_k}(x), \quad \tilde{g}_{i\bar{j},k\bar{l}}(x) = \frac{\partial^2 \tilde{g}_{i\bar{j}}}{\partial \xi_l \partial \xi_k}(x) = \frac{\partial^2 \nu_{i\bar{j}}}{\partial \xi_l \partial \xi_k}(x)$$

From this fact, and $|a| = |-(\tilde{g}_{2\bar{2}}(x))^{-1}\tilde{g}_{1\bar{2},1}(x)| \leq |\tilde{g}_{1\bar{2},1}(x)|$ we get that $|\hat{g}_{1\bar{1},1\bar{1}}(x)| \leq C$. Similarly, when $(i, j, k, l) \neq (1, 1, 1, 1)$ we have $|g_{i\bar{j},k\bar{l}}(x)| = |\hat{g}_{i\bar{j},k\bar{l}}(x)| \leq C$, and the claim follows.

Claim 6

$$T_2 \leq C + (\beta - 1)^2 |z_1|^{2\beta-4} |v_1|^2 |w_1|^2$$

Define a non-negative bilinear hermitian form on tensors $a = [a_{i\bar{j}k}]$ satisfying $a_{i\bar{j}k} = a_{k\bar{j}i}$ by

$$\langle [a_{i\bar{j}k}], [b_{p\bar{q}r}] \rangle = \sum g^{q\bar{j}}(x) (w_i a_{i\bar{j}k} v_k) \overline{(w_p b_{p\bar{q}r} v_r)}$$

Then

$$T_2 = \|D + E\|^2$$

with $D_{ijk} = \hat{g}_{i\bar{j},k}$ and $E_{ijk} = (\beta - 1)|z_1|^{2\beta-4}\bar{z}_1$ if $(ijk) = (111)$ and $E_{ijk} = 0$ otherwise. We first estimate

$$\|E\|^2 = (\beta - 1)^2 |z_1|^{4\beta-6} g^{1\bar{1}}(x) |v_1|^2 |w_1|^2$$

where $g^{1\bar{1}} = \det(g)^{-1} g_{2\bar{2}}$.

$$\det(g) = (|z_1|^{2\beta-2} + \hat{g}_{1\bar{1}})\hat{g}_{2\bar{2}} - |\hat{g}_{1\bar{2}}|^2 = \hat{g}_{2\bar{2}}|z_1|^{2\beta-2} (1 + (\hat{g}_{2\bar{2}})^{-1} \det(\hat{g})|z_1|^{2-2\beta})$$

Unwinding notation we have that at the point x , $\hat{g}_{2\bar{2}} = 1 + \epsilon\nu_{2\bar{2}}(x)$ and $\det(\hat{g})(x) = \epsilon\nu_{1\bar{1}}(x) + \epsilon^2 \det(\nu)(x)$. We conclude that $(\hat{g}_{2\bar{2}})^{-1} \det(\hat{g}) \geq Q^{-1}\epsilon$, so

$$g^{1\bar{1}}(x) \leq (1 + \delta)^{-1}|z_1|^{2-2\beta}$$

with $\delta = Q^{-1}\epsilon|z_1|^{2-2\beta}$. We get

$$\|E\|^2 \leq (1 + \delta)^{-1}(\beta - 1)^2|z_1|^{2\beta-4}|v_1|^2|w_1|^2$$

Next we do a trick

$$\|T_2\|^2 \leq (1 + \delta^{-1})\|D\|^2 + (1 + \delta)\|E\|^2$$

The claim (and the lemma) will follow if we can bound $\epsilon^{-1}|z_1|^{2\beta-2}\|D\|^2$.

$$\|D\|^2 = \sum_{s,t,i,j,k,l=1}^2 g^{s\bar{t}}(x)\hat{g}_{i\bar{t},k}(x)\hat{g}_{s\bar{j},l}(x)v_i\bar{v}_j w_k \bar{w}_l = \sum_{i,j,k,l=1}^2 g^{1\bar{1}}(x)\hat{g}_{i\bar{1},k}(x)\hat{g}_{1\bar{j},l}(x)v_i\bar{v}_j w_k \bar{w}_l$$

(The second equality follows from the first claim). Since $g^{1\bar{1}}(x) \leq |z_1|^{2-2\beta}$ and $|\hat{g}_{i\bar{j},k}(x)| \leq C\epsilon$, the estimate follows. \square

4 Linear analysis

4.1 Interior Schauder Estimates and metrics with cone singularities.

Consider the singular metric $g_{(\beta)} = \beta^2|z_1|^{2\beta-2}|dz_1|^2 + |dz_2|^2$ on \mathbb{C}^2 . We want to define Holder continuous $(1,0)$ and $(1,1)$ forms. Note that under the map $z_1 = r_1^{1/\beta} e^{i\theta_1}$ we have $g_{(\beta)} = dr_1^2 + \beta^2 r_1^2 d\theta_1^2 + |dz_2|^2$. Set $\epsilon = dr_1 + i\beta r_1 d\theta_1$. A $(1,0)$ form η is called C^α if $\eta = f_1\epsilon + f_2 dw$ with f_1, f_2 C^α functions in the usual sense in the coordinates $(r_1 e^{i\theta_1}, z_2)$. It is also required that $f_1 = 0$ on $\{z_1 = 0\}$. If we change ϵ by $\tilde{\epsilon} = e^{i\theta}\epsilon = \beta|z_1|^{\beta-1}dz_1$, say, in the definition then the vanishing condition implies that we get the same space. In order to define C^α $(1,1)$ forms we use the basis $\{\epsilon\bar{\epsilon}, \epsilon d\bar{w}, dw\bar{\epsilon}, dw d\bar{w}\}$, as above we ask the components to be C^α functions and we require the components corresponding to $\epsilon d\bar{w}, dw\bar{\epsilon}$ to vanish on the singular set. Finally we set $C^{2,\alpha}$ to be the space of C^α functions u such that $\partial u, \bar{\partial}\bar{\partial}u$ are C^α , with a choice of some obvious norm.

We are interested in the equation $\Delta u = f$ where Δ is the laplace operator of $g_{(\beta)}$. We define L_1^2 on domains of \mathbb{C}^2 by means of the usual norm $\|u\|_{L_1^2} = \int |\nabla u|^2 + \int u^2$. In the coordinates $(r_1 e^{i\theta_1}, z_2)$, $\beta^2 g_{euc} \leq g_{(\beta)} \leq (1 + \beta^2)g_{euc}$, so that L_1^2 coincides with the standard Sobolev space in these coordinates. Let u be a function that is locally in L_1^2 . We say that u is a weak solution of $\Delta u = f$ if

$$\int \langle \nabla u, \nabla \phi \rangle = - \int f \phi$$

for all smooth compactly supported ϕ .

Proposition 3 Fix $\alpha < \beta^{-1} - 1$, then there exist a constant C such that if u is a weak solution of $\Delta u = f$ on B_2 and $f \in C^\alpha(B_2)$ then $u \in C^{2,\alpha}(B_1)$ and

$$\|u\|_{C^{2,\alpha}(B_1)} \leq C (\|f\|_{C^\alpha(B_2)} + \|u\|_{C^0(B_2)}) \quad (4.1)$$

We mention 3 differences between Proposition 3 and the standard Schauder estimates

- We don't have estimates for all the second derivatives of u . (E.g. $\partial^2 u / \partial r_1^2$).

- The component of ∂u corresponding to ϵ needs to vanish along the singular set provided $\Delta u \in C^\alpha$.
- The estimates require $\alpha < \beta^{-1} - 1$.

All these facts can be explained by the fact that if p is a point outside the singular set, and $\Gamma_p = G(\cdot, p)$ with G the Green's function for Δ , then around points of $\{z_1 = 0\}$ one can write a convergent series expansion

$$\Gamma_p = \sum_{j,k \geq 0} a_{j,k}(z_2) r_1^{(k/\beta)+2j} \cos(k\theta_1)$$

with $a_{j,k}$ smooth functions.

Now let η be a $(1,1)$ form on B_2 with $\|\eta\|_{C^\alpha(B_2)} \leq \epsilon$. Assume that η has support contained in B_1 and consider the operator $Lu = \Delta u + \langle \partial \bar{\partial} u, \eta \rangle$. If $\epsilon < 1/(2C)$ we can use 4.1 to get the estimate

$$\|u\|_{C^{2,\alpha}(B_1)} \leq 2C (\|Lu\|_{C^\alpha(B_2)} + \|u\|_{C^0(B_2)}) \quad (4.2)$$

for all functions $u \in C^{2,\alpha}(B_2)$.

Now let C be our smooth curve in \mathbb{C}^2 and let ω be a (smooth) Kahler metric in the complement of C . We say that ω is a metric with cone singularities along C of angle β if around each $p \in C$ we can find holomorphic coordinates (z_1, z_2) such that

$$\omega = \omega_{(\beta)} + \eta \quad (4.3)$$

with $\eta \in C^\alpha$ and $\eta(p) = 0$.

Given our curve C and a bounded open subset U of \mathbb{C}^2 we can define the space $C^{2,\alpha}(U)$ by taking a finite cover of U with coordinates in which $C = \{z_1 = 0\}$. Let $p \in C$ and write ω as in 4.3. After a dilation and multiplying by a cut-off function we can assume that in a smaller neighborhood of p we have $\Delta_\omega = L$ with L as in 4.2. From here we get that

$$\|u\|_{C^{2,\alpha}(U)} \leq C (\|\Delta_\omega u\|_{C^\alpha(V)} + \|u\|_{C^0(V)}) \quad (4.4)$$

for all $u \in C^{2,\alpha}(V)$. In 4.4 we assume that U is compactly contained in V . The constant C depends on ω, U, V .

Finally let us say that the metrics we have constructed in the previous section have cone singularities in the sense of 4.3 because of the following

Lemma 10 *Let ω be a Kahler metric on $\mathbb{C}^2 \setminus C$ such that around each $p \in C$ we can find holomorphic coordinates (z_1, z_2) such that*

$$\omega = \Omega + i\partial\bar{\partial}(F|z_1|^{2\beta})$$

with Ω a smooth $(1,1)$ form such that $\Omega(\partial/\partial z_2, \partial/\partial \bar{z}_2)(p) > 0$ and F a smooth positive function. Then ω has cone singularities in the sense of 4.3

Proof: This follows from the computation

$$i\partial\bar{\partial}(F|z_1|^{2\beta}) = |z_1|^{2\beta} i\partial\bar{\partial}F + \beta|z_1|^{2\beta-2} (\bar{z}_1 idz_1 \bar{\partial}F + z_1 \partial F d\bar{z}_1) + \beta^2 F |z_1|^{2\beta-2} idz_1 d\bar{z}_1$$

Setting $\tilde{z}_1 = az_1$, $\tilde{z}_2 = bz_2$ with $a = F(p)^{1/2}$ and $b = (\Omega(\partial/\partial z_2, \partial/\partial \bar{z}_2)(p))^{1/2}$ one gets 4.3

□

4.2 Weighted Holder spaces

Let ω_F be the flat metric. Consider the annulus $A_1 = B_2 \setminus \overline{B_1}$, where $B_R = \{r < R\}$. We know that around each $p \in L \cap A_1$ we can find coordinates (z_1, z_2) in which $\omega_F = g_{(\beta)}$ and that ω_F is locally isometric to the euclidean metric outside L . We fix a finite cover of A_1 by such coordinates and define the space $C^{2,\alpha}(A_1)$ in the obvious way (similarly we can define the space C^α). Alternatively (in more intrinsic terms) we could have taken an orthonormal basis for the $(1,0)$ forms $\{\tau_1, \tau_2\}$, for example by applying Gram-Schmidt to $\{dz, dw\}$ over $A_1 \setminus L$, and ask for the components of ∂u and $\partial\bar{\partial}u$ with respect

to τ_i and $\tau_i \overline{\tau_j}$ respectively to be C^α . (Correspondingly we can define the space C^α by considering the distance induced by ω_F and applying the standard definition).

It follows from the standard Schauder estimates and Proposition 3 that there exist a constant C such that for every $u \in C^{2,\alpha}(\tilde{A}_1)$

$$\|u\|_{C^{2,\alpha}(A_1)} \leq C \left(\|f\|_{C^\alpha(\tilde{A}_1)} + \|u\|_{C^0(\tilde{A}_1)} \right) \quad (4.5)$$

where $\Delta u = f$ is the laplacian of u with respect to ω_F and $\tilde{A}_1 = B_4 \setminus \overline{B_{1/2}}$.

Let $\gamma \in \mathbb{R}$, we want to define the space C_γ^α . For $\lambda > 0$, denote $A_\lambda = B_{2\lambda} \setminus B_\lambda$. In other words $A_\lambda = D_\lambda(A_1)$ where D_λ is the map given in spherical coordinates by $D_\lambda(r, \theta) = (\lambda r, \theta)$. Note that in complex coordinates $D_\lambda(z, w) = (\lambda^{2/c} z, \lambda^{2/c} w)$. Let f be a continuous function on $\mathbb{C}^2 \setminus \{0\}$. Define $f_{\lambda,\gamma} = \lambda^{-\gamma} (f \circ D_\lambda)$ and think of it as a function on A_1 . Finally we set

$$\|f\|_{\alpha,\gamma} = \sup_{\lambda>0} \|f_{\lambda,\gamma}\|_{C^\alpha(A_1)} \quad (4.6)$$

It follows that if $f \in C_\gamma^\alpha$ (the space of functions in $\mathbb{C}^2 \setminus \{0\}$ for which the above norm is finite), then $|f(x)| \leq A r(x)^\gamma$ for some constant A . In fact, if we let $\|f\|_{0,\gamma} = \sup_{\lambda>0} \|f_{\lambda,\gamma}\|_{C^0(A_1)}$ we clearly have $\|f\|_{0,\gamma} \leq \|f\|_{\alpha,\gamma}$ and $\|f\|_{0,\gamma}$ is easily seen to be equivalent to $\sup_x r(x)^{-\gamma} |f(x)|$.

It is clear that if we use \tilde{A}_1 instead we would get an equivalent norm, i.e, there exist a constant C such that

$$\sup_{\lambda>0} \|f_{\lambda,\gamma}\|_{C^\alpha(\tilde{A}_1)} \leq C \|f\|_{\alpha,\gamma}$$

Having said what is the space $C^{2,\alpha}$ on A_1 we can define the space $C_\delta^{2,\alpha}$ to be the space of functions u on $\mathbb{C}^2 \setminus \{0\}$ for which

$$\|u\|_{2,\alpha,\delta} = \sup_{\lambda>0} \|u_{\lambda,\delta}\|_{C^{2,\alpha}(A_1)} \quad (4.7)$$

is finite. As above δ is any fixed real number.

With these definitions we claim that Δ defines a bounded operator from $C_\delta^{2,\alpha}$ to $C_{\delta-2}^\alpha$. Indeed, from the expression

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{3}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{\overline{g}} \quad (4.8)$$

we get that $\Delta u_\lambda = \lambda^2 (\Delta u)_\lambda$, where we denote $u_\lambda = u \circ D_\lambda$. Now take $u \in C_\delta^{2,\alpha}$, write $\Delta u = f$ and let $\lambda > 0$. Then

$$f_{\lambda,\delta-2} = \lambda^{-\delta+2} (\Delta u)_\lambda = \lambda^{-\delta} \Delta u_\lambda$$

and our claim follows from the fact that $\Delta : C^{2,\alpha}(A_1) \rightarrow C^\alpha(A_1)$ is a bounded operator.

Let us give an equivalent norm in $C_\delta^{2,\alpha}$ which will make evident the fact that if u belongs to this space then $|\partial \overline{\partial} u|_{g_F} = O(r^{\delta-2})$. In order to do this we note that on $\mathbb{C}^2 \setminus L$ we have an (up to a factor of $\sqrt{2}$) orthonormal basis (w.r.t. g_F) of the $(1, 0)$ forms given by $\{\tau_1, \tau_2\}$ (see 2.15) such that $D_\lambda^* \tau_i = \lambda \tau_i$. Given a function u we write $\partial u = \sum_i u_i \tau_i$ and $\partial \overline{\partial} u = \sum_{i,j} u_{i\overline{j}} \tau_i \overline{\tau_j}$. We claim that

$$\|u\|_{2,\alpha,\delta} = \|u\|_{0,\delta} + \sum_i \|u_i\|_{\alpha,\delta-1} + \sum_{i,j} \|u_{i\overline{j}}\|_{\alpha,\delta-2} \quad (4.9)$$

defines an equivalent norm as the one defined above. (Our claim justifies the abuse of notation since 4.9 is not exactly equal to 4.7). Since $\Delta u = u_{1\overline{1}} + u_{2\overline{2}}$ we see again that $\Delta : C_\delta^{2,\alpha} \rightarrow C_{\delta-2}^\alpha$ is a bounded map.

We compute $\|u_{\lambda,\delta}\|_{C^{2,\alpha}(A_1)}$ using the basis $\{\tau_1, \tau_2\}$. Since D_λ is holomorphic we have that $\partial u_\lambda = D_\lambda^* \partial u = \lambda \sum_i (u_i)_\lambda \tau_i$ and that $\partial \overline{\partial} u_\lambda = D_\lambda^* \partial \overline{\partial} u = \lambda^2 \sum_{i,j} (u_{i\overline{j}})_\lambda \tau_i \overline{\tau_j}$. Our claim then follows from

$$\|u_{\lambda,\delta}\|_{C^{2,\alpha}(A_1)} = \|\lambda^{-\delta} u_\lambda\|_{C^0(A_1)} + \sum_i \|\lambda^{-\delta+1} (u_i)_\lambda\|_{C^\alpha(A_1)} + \sum_{i,j} \|\lambda^{-\delta+2} (u_{i\overline{j}})_\lambda\|_{C^\alpha(A_1)}$$

. In arguments in which the Holder exponent α is not crucially needed we will say that a function is in C^2 if the components $u_{i\bar{j}}$ are continuous. Similarly we can give a definition of C_δ^2 .

We are now ready to state our first main estimate

Lemma 11 *Let $\alpha < \beta^{-1} - 1$ and $\delta \in \mathbb{R}$. Then there exist a constant $C = C(\alpha, \delta)$ such that for every $u \in C_\delta^{2,\alpha}$ with $\Delta u = f$*

$$\|u\|_{2,\alpha,\delta} \leq C(\|f\|_{\alpha,\delta-2} + \|u\|_{0,\delta})$$

Proof: Write $\delta = \gamma + 2$. Let $\lambda > 0$ we apply the interior estimate 4.5 to $u_{\lambda,\delta} = \lambda^{-\delta} u_\lambda$ to get

$$\|u_{\lambda,\delta}\|_{C^{2,\alpha}(A_1)} \leq C \left(\|\lambda^{-\delta+2} f_\lambda\|_{C^\alpha(\tilde{A}_1)} + \|\lambda^{-\delta} u_\lambda\|_{C^0(\tilde{A}_1)} \right)$$

and note that the first term on the r.h.s. is bounded by $\|f\|_{\alpha,\gamma}$ and the second term is bounded by $\|u\|_{0,\gamma+2}$. □

Remark 3 *In fact we have proved that if u is locally in $C^{2,\alpha}$ and $\|u\|_{0,\delta}$ is finite, then $u \in C_\delta^{2,\alpha}$ and the above estimate holds.*

Our next goal is to bound $\|u\|_{0,\delta}$ in terms of $\|f\|_{\alpha,\delta-2}$. It turns out that this is true, except when δ belongs to the discrete set of 'Indicial Roots'. In order to explain what is this set we digress a little and discuss some basics of spectral theory for $\Delta_{\bar{g}}$, the laplacian of the singular metric on the 3-sphere.

First we note that on (S^3, \bar{g}) there is an obvious definition of the spaces L^2 and L_1^2 . Since there is a diffeomorphism χ of $S^3 \setminus L$ such that $\chi^* \bar{g}$ is quasi-isometric to a smooth metric on S^3 we see that L^2 and L_1^2 correspond under χ to the usual spaces. In particular we have that $L_1^2 \subset L^2$ is compact.

If we write the norms as $\|f\|_{L^2}^2 = \int f^2$ and $\|u\|_{L_1^2}^2 = \int u^2 + \int |\nabla u|^2$ we see that $f \in L^2$ defines a bounded linear functional T on L_1^2 by $T(\phi) = \int f\phi$. If we write $T = \langle u, - \rangle_{L_1^2}$ then u is said to be a weak solution of $-\Delta_{\bar{g}} u + u = f$. The map $K(f) = u$ is a bounded linear map between L^2 and L_1^2 , composing this map with the compact inclusion we have a map $K : L^2 \rightarrow L^2$ which is compact and self-adjoint. It follows from the spectral theorem that we can find an orthonormal basis $\{\phi_i\}_{i \geq 0}$ of L^2 such that $K(\phi_i) = s_i \phi_i$ and $s_i \rightarrow 0$. Unwinding the definitions we get that $\Delta_{\bar{g}} \phi_i = -\lambda_i \phi_i$ with $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ and $\lambda_i = (1 - s_i)/s_i \rightarrow \infty$.

For each λ_i define δ_i^\pm to be the solutions of $\{s(s+2) = \lambda_i\}$ with δ_i^+ non-negative and δ_i^- non-positive (in fact ≤ -2). The set of Indicial Roots is set to be $I = \{\delta_i^\pm, i \geq 0\}$. With this definition we can state the following

Lemma 12 *Let $u \in C_\delta^2$ such that $\Delta u = 0$ and $\delta \notin I$. Then $u = 0$*

Proof:

Write $u(r, \theta) = \sum_{i=0}^\infty u_i(r) \phi_i(\theta)$, where $u_i(r) = \int_{S^3} u(r, \cdot) \phi_i$. It follows from Holder's inequality that if $|u| \leq Cr^\delta$ then $|u_i(r)| \leq C(\text{Vol}(\bar{g}))^{1/2} r^\delta$. On the other hand the equation $\Delta u = 0$ implies that

$$u_i'' + \frac{3}{r} u_i' - \frac{\lambda_i}{r^2} u_i = 0$$

so that $u_i = Ar^{\delta_i^+} + Br^{\delta_i^-}$ for some constants A and B . Since $\delta \neq \delta_i^\pm$ we get that $u_i = 0$. □

Proposition 4 *Let $\alpha < \beta^{-1} - 1$ and $\delta \in \mathbb{R} \setminus I$. Then there exist $C = C(\alpha, \delta)$ such that*

$$\|u\|_{2,\alpha,\delta} \leq C\|f\|_{\alpha,\delta-2} \tag{4.10}$$

for every $u \in C_\delta^{2,\alpha}$ with $\Delta u = f$

Proof: If the result was not true then we would be able to take a sequence $\|u_k\|_{2,\alpha,\delta} = 1$ with $\Delta u_k = f_k$ and $\|f_k\|_{\alpha,\delta-2} \rightarrow 0$. It follows from Lemma 11 that $\|u_k\|_{0,\delta} \geq 2\epsilon$ for some $\epsilon > 0$. Hence we can find x_k such that $r(x_k)^{-\delta} |u_k(x_k)| \geq \epsilon$. Consider the sequence $\tilde{u}_k = (u_k)_{L_k,\delta}$ where $L_k = r(x_k)$. Write $x_k = (r(x_k), \theta_k)$, then $|\tilde{u}_k(\tilde{x}_k)| \geq \epsilon$ with $\tilde{x}_k = (1, \theta_k)$. On the other hand $\tilde{f}_k = \Delta \tilde{u}_k = L_k^{-\delta+2} (f_k)_{L_k} = (f_k)_{L_k,\gamma}$.

(Where $\gamma = \delta - 2$). The key point is that $\|u\|_{2,\alpha,\delta} = \|u_{L,\delta}\|_{2,\alpha,\delta}$ and $\|f\|_{\alpha,\gamma} = \|f_{L,\gamma}\|_{\alpha,\gamma}$ for any $L > 0$ and f, g any functions. So that $\|\tilde{u}_k\|_{2,\alpha,\delta} = 1$ and $\|\tilde{f}_k\|_{\alpha,\delta-2} \rightarrow 0$. Let $K_n = \overline{B_n} \setminus B_{1/n}$ for n an integer ≥ 2 . Arzela-Ascoli and the bound $\|\tilde{u}_k\|_{2,\alpha,\delta} = 1$ imply that we can take a subsequence $\tilde{u}_k^{(n)}$ which converges in $C^2(K_n)$ to some function u_n such that $\Delta u_n = 0$. The diagonal subsequence $\tilde{u}_n^{(n)}$ converges to a function u in $C^2 \setminus \{0\}$ which is in C_δ^2 and $\Delta u = 0$. Since $|\tilde{u}_k(\tilde{x}_k)| \geq \epsilon$ we see that $u \neq 0$, but this contradicts Lemma 12

□

In practice we will only use the estimate 4.10 for functions u with support outside B_1 . For this functions we can give another equivalent definition of the norms 4.6 and 4.7.

Slightly abusing notation let us set

$$\|f\|_{\alpha,\gamma} = \|f\|_{0,\gamma} + [f]_{\alpha,\gamma-\alpha} \quad (4.11)$$

for functions f with $\text{supp}(f) \subset B_1^c$ and where

$$[f]_{\alpha,\gamma-\alpha} = \sup_{x,y} \min\{r(x), r(y)\}^{-\gamma+\alpha} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}$$

when $\gamma < 0$. (If $\gamma > 0$ we replace $\min\{r(x), r(y)\}$ by $\max\{r(x), r(y)\}$).

Claim 7 4.6 and 4.11 define equivalent norms

Proof:

We prove that 4.11 is bounded by a constant times 4.6. Consider the case of $\gamma < 0$. Take $x, y \in \mathbb{C}^2$ with $r(x) \leq r(y)$ such that

$$(1/2)[f]_{\alpha,\gamma-\alpha} \leq r(x)^{-\gamma+\alpha} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}$$

Assume first that $r(y) \geq (5/4)r(x)$, say, then $d(x,y) \geq d(y,0) - d(x,0) \geq (1/4)r(x)$ so that

$$(1/2)[f]_{\alpha,\gamma-\alpha} \leq r(x)^{-\gamma} |f(x)| + r(x)^{-\gamma} |f(y)|$$

when $\gamma < 0$, $r(x)^{-\gamma} |f(y)| \leq r(y)^{-\gamma} |f(y)|$ and this last term is bounded by 4.6. When $r(y) \leq (5/4)r(x)$ we write $x = (r(x), \theta)$ and $y = (r(y), \psi)$. Let $\tilde{x} = (3/2, \theta)$ and $\tilde{y} = (\frac{3r(y)}{2r(x)}, \psi)$. Set $\lambda = (2/3)r(x)$ so that $D_\lambda(\tilde{x}) = x$ and $D_\lambda(\tilde{y}) = y$. Note that $\tilde{x}, \tilde{y} \in A_1$ ($r(\tilde{y}) \leq 15/8 < 2$), so that 4.6 gives us a bound for

$$\lambda^{-\gamma} \frac{|f(x) - f(y)|}{d(\tilde{x}, \tilde{y})^\alpha} = (2/3)^{-\gamma} r(x)^{-\gamma+\alpha} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}$$

From this we get that 4.11 is bounded by a constant times 4.6. The reverse inequality follows similarly. □

Putting 4.11 and 4.9 together we get that the norm 4.7 we defined is equivalent (for functions u with $\text{supp}(u) \subset B_1^c$) to the commonly used [8] [4].

Finally let us point that $(-2, 0) \cap I = \phi$ independently of \bar{g} . In fact, for this range one can give an alternative proof of 4.10 which does not use the spectrum of $\Delta_{\bar{g}}$.

Lemma 13 Let $u \in C^2$ with $\text{supp}(u) \subset B_1^c$. Assume $\Delta u = f \in C_{\delta-2}^0$ for some $\delta \in (-2, 0)$ and that $u \in C_\mu^0$ for some $\mu < 0$. Then

$$\|u\|_{0,\delta} \leq c_\delta \|f\|_{0,\delta-2}$$

with $c_\delta = -(\delta + 2)^{-1} \delta^{-1}$.

Proof:

From 4.8 we have that $\Delta r^\delta = (\delta + 2)\delta r^{\delta-2} = -Q_\delta r^{\delta-2}$ with $Q_\delta = -(\delta + 2)\delta > 0$. On $U_R = B_R \setminus B_1$ consider the function $h = u - Ar^\delta - m_R$ where $m_R = \sup_{\partial B_R} u$ and $A = \|f\|_{0,\delta-2}/Q_\delta$. Then

$$\Delta h = f + \|f\|_{0,\delta-2} r^{\delta-2} \geq 0$$

$h \leq 0$ on ∂B_1 since u has support outside B_1 . By our choice of m_R , $h \leq 0$ on ∂B_R . The maximum principle implies that $h \leq 0$ in U_R , i.e. for every $x \in U_R$ we have that

$$u(x) \leq (\|f\|_{0,\delta-2}/Q_\delta)r(x)^\delta + m_R$$

. Since $u \in C_\mu^0$ for some $\mu < 0$ we have that $\lim_{R \rightarrow \infty} m_R = 0$. We let $R \rightarrow \infty$ and get the desired upper bound on u . The lower bound, and hence the lemma, follows by applying the upper bound to $-u$. \square

Let us explain how one can use the maximum principle in this context. Let $A = B_{R_2} \setminus B_{R_1} \subset \mathbb{C}^2 \setminus \{0\}$. Let $h \in C^2(A)$ be such that $\Delta h \geq 0$ and $h|_{\partial A} \leq 0$. We claim that $h \leq 0$ on A , if this was not the case we can find $p \in A$ such that $h(p) = \sup_A h = 2m > 0$. If $p \notin L$ this would contradict the usual maximum principle. Then $p \in L$. Let $\epsilon < \beta$ and δ be small enough such that $\delta|l|^{2\epsilon} \leq m$ on ∂A . Consider the function $H = h + \delta|l|^{2\epsilon}$. By our choices H has a local maximum at some point $q \in A$. Since $i\partial\bar{\partial}|l|^{2\epsilon} \geq 0$ we still have $\Delta H \geq 0$. Since $\epsilon < \beta$ and h is a C^1 function, we have that $q \notin L$, contradicting the usual maximum principle. In fact this argument can be adapted to other situations. For example the same holds if h is C^α , smooth outside L with $\Delta h \geq 0$ (one then needs to take $\epsilon < \alpha\beta$).

4.3 Main result

In this subsection we study the mapping properties (between weighted spaces) of the laplacian of a metric ω with cone singularities along C asymptotic to ω_F . We fix such an ω given by Lemma 6.

We want to define our weighted Holder spaces. The notation is the one of subsection 3.1. Fix N large enough such that $C \cap B_N^c \subset U_{2R,\delta/2}$. Let χ be a smooth function equal to 1 on B_{N+1}^c which vanishes on B_N . For a function $u : \mathbb{C}^2 \rightarrow \mathbb{R}$ we write $u_\infty = \chi u \circ G$. We change notation and introduce a $'$ on the norms of the previous subsection. The space $C_\delta^{2,\alpha}$ (C_γ^α) is defined to be the set of functions u (f) such that the norm

$$\|u\|_{2,\alpha,\delta} = \|u\|_{C^{2,\alpha}(B_{N+1})} + \|u_\infty\|'_{2,\alpha,\delta} \quad (4.12)$$

$$\|f\|_{\alpha,\gamma} = \|f\|_{C^\alpha(B_{N+1})} + \|f_\infty\|'_{\alpha,\gamma} \quad (4.13)$$

is finite. The fact is that these are Banach spaces.

Write Δ for the laplacian of ω . We apply the estimates of the previous two subsections to get the following

Corollary 1 *Let $\delta \notin I$ and $\alpha < \beta^{-1} - 1$. Then there exist a compact set K and a constant C such that for all $u \in C_\delta^{2,\alpha}$ with $\Delta u = f$ we have*

$$\|u\|_{2,\alpha,\delta} \leq C (\|u\|_{C^0(K)} + \|f\|_{\alpha,\delta-2}) \quad (4.14)$$

Proof:

The key point is that if $v \in (C_\delta^{2,\alpha})'$ with support on B_L^c ,

$$\|\Delta_{G^*g}v - \Delta_F v\|'_{\alpha,\delta-2} \leq c_L \|v\|'_{2,\alpha,\delta} \quad (4.15)$$

with $c_L \rightarrow 0$ as $L \rightarrow \infty$. Where g is the metric corresponding to ω and Δ_F is the laplacian of the flat metric. Since $G^*g = g_F$ in a region $U_{\delta',R'}$ and $|G^*g - g_F|_{g_F} = O(r^\mu)$ for some $\mu < 0$ with derivatives on the complement of $U_{\delta',R'}$, 4.15 holds.

The lemma then follows from 4.10 and the interior estimates. \square

Lemma 14 $\Delta : C_\delta^{2,\alpha} \rightarrow C_{\delta-2}^\alpha$ has finite dimensional kernel for any δ and closed image when $\delta \notin I$.

Proof:

Let us start by proving the statement about the kernel. Assume first that $\delta \notin I$ and let $u_k \in C_\delta^{2,\alpha}$ with $\Delta u_k = 0$ and $\|u_k\|_{2,\alpha,\delta} = 1$. By Arzela-Ascoli we can take a subsequence which converges in $C^0(K)$ to some function. We apply the estimate 4.14 to conclude that the subsequence is Cauchy in $C_\delta^{2,\alpha}$ and hence $\ker(\Delta)$ is finite dimensional. In the case that $\delta \in I$ just take $\tilde{\delta} > \delta$, $\tilde{\delta} \notin I$ and note that $C_\delta^{2,\alpha} \subset C_{\tilde{\delta}}^{2,\alpha}$.

To prove that the image is closed let us write $C_\delta^{2,\alpha} = V \oplus \ker(\Delta)$ for some closed subspace V . We claim that there exist a constant C such that $\|u\|_{2,\alpha,\delta} \leq C\|f\|_{\alpha,\delta-2}$ for every $u \in V$. If this was not true then we would get a sequence such that $\|u_k\|_{2,\alpha,\delta} = 1$ and $\|f_k\|_{\alpha,\delta-2} \rightarrow 0$. It follows from Arzela-Ascoli and 4.14 that, after taking a subsequence, we can assume that u_k converges in $C_\delta^{2,\alpha}$ to some function u with $\Delta u = 0$. Since $u \in V$ then $u = 0$ and this contradicts $\|u_k\|_{2,\alpha,\delta} = 1$. Finally let $f_k = \Delta u_k$ with $f_k \rightarrow f$ in $C_{\delta-2}^\alpha$. We can assume that $u_k \in V$. The estimate we just proved implies that $\{u_k\}$ is Cauchy and converges to some $u \in C_\delta^{2,\alpha}$ with $\Delta u = f$. \square

Let \mathcal{H} be the completion of the space of compactly supported functions $\phi \in L_1^2$ under the Dirichlet norm $\int |\nabla \phi|^2$. (In a more precise notation we should write $\int_{\mathbb{C}^2} |\nabla^\omega \phi|^2$).

Lemma 15 (Sobolev inequality) *There exist C such that*

$$\left(\int |\phi|^4 \right)^{1/2} \leq C \int |\nabla \phi|^2 \quad (4.16)$$

for every $\phi \in \mathcal{H}$

Proof: This follows since we can find a diffeomorphism of $\mathbb{C}^2 \setminus C$ under which ω is quasi-isometric to the euclidean metric. \square

Let $f \in L^{4/3}$. It follows from 4.16 that $T_f(\phi) = \int f\phi$ defines a bounded functional on \mathcal{H} . A weak solution of $\Delta u = f$ is a function $u \in \mathcal{H}$ such that $-\int \langle \nabla u, \nabla \phi \rangle = \int f\phi$ for every $\phi \in \mathcal{H}$. It follows from Proposition 3 that if f is locally in C^α then u is locally in $C^{2,\alpha}$.

Lemma 16 *Let $f \in C_c^\alpha$ and $u \in \mathcal{H}$ a weak solution of $\Delta u = f$. Then $u \in C_\delta^{2,\alpha}$ for any $\delta > -2$*

Proof:

Let

$$\|u\|_{L_\delta^2}^2 = \int |u|^2 \rho^{-2\delta} \rho^{-4}$$

Since $u \in \mathcal{H}$ we get that $\int |u|^4$ is finite (in fact it is bounded by $\|f\|_{L^{4/3}}$). From Holder's inequality we have that

$$\|u\|_{L_\delta^2}^2 \leq \left(\int |u|^4 \right)^{1/2} \left(\int \rho^{-4(\delta+2)} \right)^{1/2}$$

. If $\delta > -1$ we conclude that $\|u\|_{L_\delta^2}$ is finite.

In the interior Schauder estimates one can replace the C^0 norm in the r.h.s by the L^2 norm. Using the interior estimates in this form one gets that if u is locally in $C^{2,\alpha}$ and $\|u\|_{L_\delta^2}$ is finite, then $u \in C_\delta^{2,\alpha}$ and

$$\|u\|_{2,\alpha,\delta} \leq C \left(\|f\|_{\alpha,\delta-2} + \|u\|_{L_\delta^2} \right)$$

Hence $u \in C_\delta^0$ for any $\delta > -1$. One can then use Lemma 13 to show that in fact this is true for any $\delta > -2$ \square

Proposition 5 $\Delta : C_\delta^{2,\alpha} \rightarrow C_{\delta-2}^\alpha$ is an isomorphism when $\delta \in (-2, 0)$ and is surjective when $\delta \in (0, 2) \setminus I$.

Proof:

The fact that Δ is injective when $\delta < 0$ follows from the maximum principle or by integration by parts. The key is to prove that the map is onto. By lemma 14 it is enough to prove that the image is dense. We know from lemma 16 that the space of C^α functions with compact support is contained in the image, one detail is that this space is not dense in $C_{\delta-2}^\alpha$. But this can be overcome as follows: Take $f \in C_{\delta-2}^\alpha$ and $\delta < \tilde{\delta} < 2$ with $[\delta, \tilde{\delta}] \cap I = \emptyset$. Let h_n be a sequence of smooth cut-off functions with $h_n = 1$ on B_n and $h_n = 0$ on B_{n+1}^c . The sequence of functions $f_n = h_n f \rightarrow f$ in $C_{\delta-2}^\alpha$ so that we can

find $u \in C_{\tilde{\delta}}^{2,\alpha}$ with $\Delta u = f$. It follows from the proof of lemma 14 that we can take $u \in C_{\delta'}^{2,\alpha}$ for any $\delta' \in (\delta, \tilde{\delta}]$ and with $\|u\|_{2,\alpha,\delta'} \leq C\|f\|_{\alpha,\delta}$ with C independent of δ' . By taking the limit as $\delta' \rightarrow \delta$ we get that $u \in C_{\delta}^{2,\alpha}$ □

Remark 4 Let $\omega_u = \omega + i\partial\bar{\partial}u$ be a Kahler metric on $\mathbb{C}^2 \setminus C$ with $u \in C_{\delta}^{2,\alpha}$ for some $\delta < 2$. Then Proposition 5 holds for the laplacian of ω_u

Finally we mention some properties of these weighted spaces that will be useful to us.

- Multiplication gives a bounded map

$$C_{\gamma_1}^{\alpha} \times C_{\gamma_2}^{\alpha} \rightarrow C_{\gamma_1+\gamma_2}^{\alpha}$$

- Let $\{f_j\}_{j=1}^{\infty} \subset C_{\gamma}^{\alpha}$ with $\|f_j\|_{\alpha,\gamma} \leq C$ for some constant C . Then, after taking a subsequence, we can assume that $f_j \rightarrow f$ uniformly in compact subsets to some function f . Moreover $f \in C_{\gamma}^{\alpha}$ and $\|f\|_{\alpha,\gamma} \leq C$.
- Let $f \in C_{\tilde{\gamma}}^{\tilde{\alpha}}$ and $\alpha < \tilde{\alpha}$, $\tilde{\gamma} < \gamma$. Then for every $\epsilon > 0$ we can find $h \in C_c^{\infty}$ such that $\|f - h\|_{\alpha,\gamma} < \epsilon$.

4.4 Application

Proposition 6 There exist $u_0 \in C_{\delta}^{2,\alpha}$ for some $\delta < 2$ such that $\omega_0 = \omega + i\partial\bar{\partial}u_0$ is a Kahler form on $\mathbb{C}^2 \setminus C$ with

$$\omega_0^2 = e^{-f_0}|P|^{2\beta-2}\Omega \wedge \bar{\Omega}$$

and $f_0 \in C_c^{\infty}$

Proof:

Write

$$\omega^2 = e^{-f}|P|^{2\beta-2}\Omega \wedge \bar{\Omega}$$

We claim that there exists $0 < \tilde{\alpha} < \beta^{-1} - 1$ and $\tilde{\gamma} < 0$ such that $f \in C_{\tilde{\gamma}}^{\tilde{\alpha}}$. The fact that $f \in C_{\tilde{\gamma}}^{\tilde{\alpha}}$ on compacts subsets follows from the expression 4.3. Lemma 7 then proves the claim. (We can take any $\tilde{\gamma} > -2/c$).

Let $0 < \alpha < \tilde{\alpha}$ and $\tilde{\gamma} < \gamma < 0$ such that $\delta = \gamma + 2 \notin I$. Then there exist $\{h_j\}_{j=1}^{\infty} \subset C_c^{\infty}$ such that $\lim_{j \rightarrow \infty} \|f - h_j\|_{\alpha,\gamma} = 0$.

Consider the bounded map $\mathcal{F} : U \subset C_{\delta}^{2,\alpha} \rightarrow C_{\delta-2}^{\alpha}$ defined in a neighborhood of 0 and given by

$$\mathcal{F}(u) = \log \frac{(\omega + i\partial\bar{\partial}u)^2}{\omega^2}$$

. So that $\mathcal{F}(0) = 0$ and $D\mathcal{F}|_0 = \Delta$. By the implicit function theorem we can solve $\mathcal{F}(u_0) = f - h_N$ for some $N \gg 1$ and we get that

$$(\omega + i\partial\bar{\partial}u_0)^2 = e^{f-h_N}\omega^2$$

and the proposition is proved with $f_0 = h_N$. □

5 A priori estimates for the Monge-Ampere equation

Let ω and ω_0 be as before, recall that

$$\omega_0^2 = e^{-f_0}|P|^{2\beta-2}\Omega \wedge \bar{\Omega}$$

with $f_0 \in C_c^{\infty}$.

We fix $0 < \alpha < \beta^{-1} - 1$ and $-2 < \delta < 0$. The main result of this section is the following

Proposition 7 *There exist a constant C independent of $t \in [0, 1]$ such that if $u_t \in C_\delta^{2,\alpha}$ solves*

$$(\omega_0 + i\partial\bar{\partial}u_t)^2 = e^{tf_0}\omega_0^2 \quad (5.1)$$

then $\|u_t\|_{2,\alpha,\delta} \leq C$

In the following subsections we simplify notation and write $f = tf_0$ and $u = u_t$

5.1 C^0 estimate

We use Moser iteration. Note that $u \in C_\delta^0$ implies that $u \in L^p$ for p large and $\|u\|_0 = \lim_{p \rightarrow \infty} \|u\|_{L^p}$.

Lemma 17 *For all $p > 2$ with $p\delta + 2 < 0$, if we write $\phi = u|u|^{p/2-1}$ then we have*

$$\int_{\mathbb{C}^2} |\nabla \phi|^2 \omega_0^2 \leq p \int_{\mathbb{C}^2} u|u|^{p-2} (1 - e^f) \omega_0^2 \quad (5.2)$$

Proof: Write

$$\eta = u|u|^{p-2} i\bar{\partial}u \wedge (\omega_0 + \omega_u)$$

which is smooth on $\mathbb{C}^2 \setminus C$. Consider the region $U = B_R \setminus \{|P| \leq \epsilon\}$. By Stokes' theorem $\int_U d\eta = \int_{\partial U} \eta$ where

$$d\eta = (p-1)|u|^{p-2} i\partial u \wedge \bar{\partial}u \wedge (\omega_0 + \omega_u) + u|u|^{p-2} (e^f - 1) \omega_0^2$$

For R fixed we let $\epsilon \rightarrow 0$. Write $C_\epsilon = \{|P| = \epsilon\} \cap B_R$. Note that $\lim_{\epsilon \rightarrow 0} \text{Area}_{g_0}(C_\epsilon) = 0$ and that $|\eta|_{g_0}$ is bounded. We conclude that we can take $U = B_R$ and $\partial U = S_R$. Now note that $\text{Vol}_{g_0}(S_R) \leq CR^3$ and $|\eta|_{g_0} \leq CR^{(p-1)\delta+\delta-1}$ on S_R . The choice $p\delta < -2$ gives $\lim_{R \rightarrow \infty} \int_{S_R} \eta = 0$ and we get $\int_{\mathbb{C}^2} d\eta = 0$.

The lemma follows from $i\partial u \wedge \bar{\partial}u \wedge \omega_0 = |\nabla u|^2 \omega_0^2$, $|\nabla \phi|^2 = (p^2/4)|u|^{p-2} |\nabla u|^2$ and $i\partial u \wedge \bar{\partial}u \wedge \omega_u = F\omega_0^2$ with $F = |\nabla u|_{g_u}^2 (\omega_u^2/\omega_0^2) \geq 0$. □

Now we are ready to prove the C^0 bound on u .

Proof: The Sobolev inequality for the metric ω_0 tells us

$$\left(\int_{\mathbb{C}^2} |\phi|^4 \omega_0^2 \right)^{1/2} \leq C \int_{\mathbb{C}^2} |\nabla \phi|^2 \omega_0^2 \quad (5.3)$$

Apply this to $\phi = u|u|^{p/2-1}$, use 5.2 to get

$$\|u\|_{L^{2p}}^p \leq Cp \|u\|_{L^{p-1}}^{p-1} \quad (5.4)$$

The next step is to estimate $\|u\|_{L^{p_1}}$ for some $p_1 > 2$. In order to do this we fix some $p_0 > 2$ such that $p_0\delta + 2 < 0$ and use 5.2, 5.3 to get

$$\left(\int_{\mathbb{C}^2} |u|^{2p_0} \omega_0^2 \right)^{1/2} \leq p_0 \int_{\mathbb{C}^2} |1 - e^f| |u|^{p_0-1} \omega_0^2$$

Let $r > 1$ be given by $r(p_0 - 1) = 2p_0$ and q by $r^{-1} + q^{-1} = 1$. We replace $|1 - e^f| \leq C\rho^\gamma$. (With $\gamma = \delta - 2$). From the choices it follows that $\|\rho^\gamma\|_{L^q} \leq C$. Holder's inequality then implies that $\|u\|_{L^{p_1}} \leq C$ with $p_1 = 2p_0$.

Using the bound on $\|u\|_{L^{p_1}}$, 5.4 and an induction argument one gets a uniform bound (independent of p) on $\|u\|_{L^p}$, finally one gets $\|u\|_{C^0} = \lim_{p \rightarrow \infty} \|u\|_{L^p} \leq C$. □

5.2 C^2 estimate

We use the maximum principle.

Proposition 8 (*Chern-Lu inequality*) *Let ω and η be two Kahler metrics such that $\text{Ric}(\omega) \geq -B_1\eta$ and $\text{Bisec}(\eta) \leq B_2$ for some $B_1, B_2 > 0$. Set $\phi = \text{tr}_\omega(\eta)$. Then*

$$\Delta_\omega \log \phi \geq -B\phi \quad (5.5)$$

where $B = B_1 + B_2$.

We will use this in the complement of the curve, with $\eta = \omega_{ref}$. Note that $\text{Ric}(\omega) = (1-t)\text{Ric}(\omega_0)$. Write $\omega = \omega_{ref} + i\partial\bar{\partial}v$ (note that u and v differ by a fixed function), taking the trace w.r.t. ω we get $2 = \phi + \Delta_\omega v$. Consider the function $H = \log \phi - Av$, with $A = B + 1$. It is enough to show that H is bounded above by a uniform constant. Since $H(y) \rightarrow \log 2$ as $y \rightarrow \infty$, we can assume that H attains its global maximum at $x \in \mathbb{C}^2$. If $x \notin C$

$$0 \geq \Delta_\omega H(x) \geq -B\phi - A\Delta_\omega v = \phi(x) - 2A$$

from where we get the desired estimate.

If $x \in C$ we can assume $H(x) \geq \log 2 + 3$ and take $R > 0$ so that $H|_{\partial B_R} \leq \log 2 + 1$. Fix some $0 < \epsilon < \beta$ and consider the function $\tilde{H} = H + (1/N)|P|^{2\epsilon}$ where $N > 0$ is big enough such that $(1/N)|P|^{2\epsilon} \leq 1$ on ∂B_R . By our choices $\max_{y \in \overline{B_R}} \tilde{H} = \tilde{H}(\tilde{x})$ with $\tilde{x} \notin \partial B_R$. Since $H \in C^\alpha$ and $\epsilon < \beta$, we have that $\tilde{x} \notin C$, hence

$$0 \geq \Delta_\omega \tilde{H}(\tilde{x}) = \Delta_\omega H + (1/N)\Delta_\omega |P|^{2\epsilon} \geq \Delta_\omega H(\tilde{x}) \geq \phi(\tilde{x}) - 2A$$

We used that $\Delta_\omega |P|^{2\epsilon} \geq 0$ since $i\partial\bar{\partial}|P|^{2\epsilon} \geq 0$. Note that $H(x) \leq \tilde{H}(x) \leq \tilde{H}(\tilde{x})$ to get the estimate.

5.3 $C^{2,\alpha}$ estimate

What we want follows directly from the following

Proposition 9 [3] *Let $\phi \in C^{2,\alpha}(B_2)$ such that $\omega = \omega_{(\beta)} + i\partial\bar{\partial}\phi$ solves*

$$\omega^2 = e^f \omega_{(\beta)}^2$$

with $Q^{-1}\omega_{(\beta)} \leq \omega \leq Q\omega_{(\beta)}$, $\|\phi\|_{C^0(B_2)} \leq Q$ for some $Q > 0$. Then there exist C depending only on Q such that

$$[\partial\bar{\partial}\phi]_{C^\alpha(B_1)} \leq C$$

5.4 Weighted estimates

Let us fix some $\delta < \mu < 0$.

Claim 8 $\|u\|_{C_\mu^0} \leq C$

In order to prove this claim we introduce the norm

$$\|u\|_{L_\mu^p}^p = \int_{\mathbb{C}^2} |u|^p \rho^{-p\mu} \rho^{-4} \omega_0^2$$

Because $u \in C_\delta^0$ and $\delta < \mu$ we have that $u \in C_\mu^0$, $u \in L_\mu^p$ for all $p \geq 1$ and $\|u\|_{C_\mu^0} = \lim_{p \rightarrow \infty} \|u\|_{L_\mu^p}$.

Lemma 18 *For $p \geq 2$, $p\mu \leq -2$ we have*

$$\|u\|_{L_\mu^{2p}}^p \leq Cp \left(\|u\|_{L_\mu^{p-1}}^{p-1} + \|u\|_{L_\mu^p}^p \right) \quad (5.6)$$

To prove the lemma it is necessary a bound on $\|\partial\bar{\partial}u\|_{C^0}$. The proof is similar to the one of 5.4 and can be found in Joyce's book. To bound $\|u\|_{0,\mu}$ we start by noting that if $p_0 = (-4/\mu)$ then $\|u\|_{L^{p_0}} = \|u\|_{L^{p_0}}$ and we already have a bound on this quantity. Finally an induction argument using 5.6 gives the desired estimate.

Claim 9 $\|u\|_{2,\alpha,\delta} \leq C$

Proof:

From $\omega^2 = e^f \omega_0^2$ we have that $i\partial\bar{\partial}u \wedge (2\omega_0 + i\partial\bar{\partial}u) = (e^f - 1)\omega_0^2$ and we get

$$\Delta_0 u = (e^f - 1) + \psi \quad (5.7)$$

with $\psi = u_{i\bar{j}}^2$. We could also have written

$$\Delta u = H.(e^f - 1) \quad (5.8)$$

where Δ is the laplace operator of the metric $\omega_{u/2} = \omega_0 + i\partial\bar{\partial}(u/2)$ and $H = \omega_{u/2}^2/\omega_0^2$.

Note that $\omega_{u/2} = (1/2)\omega_0 + (1/2)\omega_u \geq (1/2)\omega_0$ and the $C^{2,\alpha}$ bound on u allow us to conclude that

$$\|u\|_{C^{2,\alpha}(B_1(x))} \leq C (\|\Delta u\|_{C^{2,\alpha}(B_2(x))} + \|u\|_{C^0(B_2(x))}) \quad (5.9)$$

with a constant C independent of x . We multiply 5.9 by $\rho(x)^{-\mu}$ and we get

$$\|u_{i\bar{j}}\|_{0,\mu} \leq C, \quad \rho(x)^{-\mu} \frac{|u_{i\bar{j}}(x) - u_{i\bar{j}}(y)|}{d(x,y)^\alpha} \leq C \text{ whenever } d(x,y) < 1 \quad (5.10)$$

Now we take $\mu < \tilde{\mu} < 0$, $\tilde{\mu} = \mu + \alpha$ such that $2\tilde{\mu} < -2$. (We started with $-2 < \delta < -1 - \alpha$, then we take $\delta < \mu < -1 - \alpha$ so that $\tilde{\mu} < -1$). We claim that 5.10 implies that $\|u_{i\bar{j}}\|_{\alpha,\tilde{\mu}}$. In fact one only needs to consider the case of $d(x,y) \geq 1$, let's say that $\rho(x) \leq \rho(y)$ and estimate

$$\rho(x)^{-\tilde{\mu}+\alpha} \frac{|u_{i\bar{j}}(x) - u_{i\bar{j}}(y)|}{d(x,y)^\alpha} \leq \rho(x)^{-\mu} (|u_{i\bar{j}}(x)| + |u_{i\bar{j}}(y)|) \leq 2C$$

We use 5.7 to conclude that $\|u\|_{2,\alpha,2+2\tilde{\mu}} \leq C$. So that $\|u_{i\bar{j}}\|_{\alpha,2\tilde{\mu}} \leq C$, from here $\|\psi\|_{\alpha,4\tilde{\mu}}$. Since $4\tilde{\mu} < -4 < \delta - 2$ we use can 5.7 again to obtain $\|u\|_{2,\alpha,\delta} \leq C$. \square

This claim finishes the proof of Proposition 7 and hence of Theorem 1. In the statement of 1 one can take any $\gamma > \max\{-2/c, -4\}$.

6 Context

6.1 Energy of the metrics

An interesting question is to ask about the energy $E = \frac{1}{8\pi^2} \int |Rm|^2$ of g_{RF} . We have the following

Conjecture 1

$$E = 1 + (\beta - 1)\chi(C) - \frac{\text{Vol}(\bar{g})}{2\pi^2} \quad (6.1)$$

We recall that \bar{g} is the corresponding singular metric on the 3-sphere and we know that $\text{Vol}(\bar{g}) = (\pi^2/2)c^2$ (see Remark 1). The Euler characteristic of C is given by $\chi(C) = 2 - 2g - d$, where $g = (d-1)(d-2)/2$ (by the degree-genus formula). Putting these facts together we obtain a formula for E which only involves d and β .

One can guess 6.1 from a more general Gauss-Bonnet type formula in the context of metrics with cone singularities. In the case $d = 2$ we have that $\chi(C) = 0$ and $\text{Vol}(\bar{g}) = 2\beta^2\pi^2$ so that 6.1 reads $E = 1 - \beta^2$. One can prove this formula by direct computation sine the metric is given by the Gibbons-Hawking ansatz.

Next we give an example with which we expect to illustrate the interest of 6.1. In \mathbb{CP}^2 with homogeneous coordinates $[x_0, x_1, x_2]$ consider the family of elliptic curves

$$C_\epsilon = \{x_0x_1x_2 - \epsilon(x_0^3 + x_1^3 + x_2^3) = 0\}$$

These curves are smooth when $\epsilon > 0$ and C_0 is the union of three lines. Fix $0 < \beta < 1$ as before. It is a (difficult) fact that for each $\epsilon \neq 0$ there exists a Kahler metric g_ϵ on $\mathbb{CP}^2 \setminus C_\epsilon$ with cone angle $2\pi\beta$ along C_ϵ and constant positive Ricci curvature on the complement of the curve, let's say $\text{Ric}(g_\epsilon) = g_\epsilon$.

Take a decreasing sequence of positive numbers $\epsilon_j \rightarrow 0$. For different values of the parameter ϵ the curves C_ϵ are different complex tori. The metrics g_{ϵ_j} are pairwise non-isometric. Denote by d_ϵ the distance induced by g_ϵ . It follows from Gromov's compactness theorem that there exist a metric space (X, d) such that $(\mathbb{CP}^2, d_{\epsilon_j}) \rightarrow (X, d)$ in the Gromov-Hausdorff sense, after taking a subsequence if necessary.

In fact there is a natural candidate for (X, d) . The S^1 action $e^{i\theta}(x_0, x_1, x_2) = (e^{i\theta}x_0, e^{i\theta}x_1, e^{i\theta}x_2)$ preserves the metric of $(\mathbb{C}_\beta)^3$. Taking an appropriate Kahler quotient we get a Kahler metric g_0 on \mathbb{CP}^2 with cone angle $2\pi\beta$ along C_0 and $\text{Ric}(g_0) = g_0$ on the complement of C_0 . When $\beta = 1/k$ the metric g_0 is (up to a constant factor) the push forward of the Fubini-Study metric under the map $[x_0, x_1, x_2] \rightarrow [x_0^k, x_1^k, x_2^k]$. The metric g_0 induces a distance d_0 and our candidate for (X, d) is (\mathbb{CP}^2, d_0) .

Let $C \subset X$ be a smooth complex curve in a compact complex surface. If g is an Einstein metric with cone angle $2\pi\beta$ (in a suitable sense) along C then the energy of g is given by

$$E = \chi(X) + (\beta - 1)\chi(C) \quad (6.2)$$

where χ denotes the Euler characteristic (see [12]). We apply this formula to the metrics g_ϵ to obtain $E(g_\epsilon) = 3$. On the other hand the energy of the metric g_0 can be computed directly (in the case when $\beta = 1/k$ it is $1/k^2$ times the energy of the Fubini-Study metric) and is given by $E(g_0) = 3\beta^2$. If our conjecture that $(\mathbb{CP}^2, d_{\epsilon_j}) \rightarrow (\mathbb{CP}^2, d_0)$ is true, then we are losing an amount of energy equal to

$$E(g_\epsilon) - E(g_0) = 3 - 3\beta^2 = 3(1 - \beta^2) \quad (6.3)$$

Let p denote any of the points $[1, 0, 0]$, $[0, 1, 0]$ or $[0, 0, 1]$ and write $\lambda_j = |\text{Rm}(g_{\epsilon_j})|(p)$. Let g_{RF} be the metric in Theorem 1 when $C = \{zw = 1\}$ and write $a = |\text{Rm}(g_{RF})|(0)$. We expect that $\lambda_j \rightarrow \infty$ and that $(\mathbb{CP}^2, \lambda_j g_{\epsilon_j}, p) \rightarrow (\mathbb{C}^2, ag_{RF}, 0)$ in the pointed Gromov-Hausdorff sense.

Alternatively, consider the embedding of \mathbb{C}^2 into \mathbb{CP}^2 given by $(u, v) \rightarrow [u, v, 1]$. In these coordinates the point $p = [0, 0, 1]$ corresponds to 0 and

$$C_\epsilon = \{uv = \epsilon(u^3 + v^3 + 1)\}$$

Write $u = \sqrt{\epsilon}z$ and $v = \sqrt{\epsilon}w$ so that $C_\epsilon = \{zw = \epsilon^{3/2}z^3 + \epsilon^{3/2}w^3 + 1\}$. Write $(u, v) = F_\epsilon(z, w)$. We can omit the discussion above on convergence of metric spaces and say that we expect that $F_\epsilon^* g_\epsilon \rightarrow g_{RF}$ (up to a constant factor) as $\epsilon \rightarrow 0$ in the sense of tensors. We see that 6.3 matches with 6.1. (Note that $E(ag_{RF}) = E(g_{RF})$ since the energy is scale invariant).

6.2 Further Research

An interesting project is to extend Theorem 1 to the case of curves for which the asymptotic lines don't need to be different. Let us consider the example of $C = \{w = z^2\}$. In this case we think that for any $1/2 < \beta < 1$ there should be a Ricci-flat metric with cone angle $2\pi\beta$ along C asymptotic to the cone $\mathbb{C}_\gamma \times \mathbb{C}$ with $\gamma = 2\beta - 1$. (A way to work out this relation between β and γ is to cut two disjoint wedge shaped regions of angle $2\pi(1 - \beta)$ from the plane, identify the corresponding sides to get a space with two cone singularities of angle $2\pi\beta$ and then let the singular points come together.) Moreover formula 6.1 allows us to compute

$$E = 1 + (\beta - 1) - \gamma = 1 - \beta \quad (6.4)$$

We expect to find these metrics in the situation of $C_\epsilon \rightarrow 2C_0$, where C_0 is now a smooth curve. Let us illustrate our speculations with an example, coming from a classical discussion involving Riemann surfaces of genus 3. (See Chapter 12 in [6].)

Let Q be a non-degenerate quadratic form in three variables, so $C_0 = \{Q = 0\} \subset \mathbb{CP}^2$ is a smooth conic. Let F be a generic polynomial of degree 4 and let $C_\epsilon = \{Q^2 + \epsilon F\} = 0$. Write $Z = \{F = 0\}$, so

that for a typical F the intersection $Z \cap C_0$ consists of 8 distinct points p_1, \dots, p_8 . For small and non-zero ϵ the curve C_ϵ is smooth and one can think of it as an approximate double cover of C_0 branched over the points p_1, \dots, p_8 . Fix some $\beta > 1/2$, assume that there exist KE metrics ω_ϵ with cone angle $2\pi\beta$ along C_ϵ and a KE metric ω_0 with cone angle $2\pi\gamma$ along C_0 . In this situation we would expect that $\omega_\epsilon \rightarrow \omega_0$. We can compute the energy of the metrics using 6.2

$$E(\omega_\epsilon) = 3 + (\beta - 1)\chi(C_\epsilon) = 3 + (\beta - 1)(-4) = 7 - 4\beta$$

$$E(\omega_0) = 3 + (\gamma - 1)\chi(C_0) = 3 + (2\beta - 2)2 = 4\beta - 1$$

The total amount of energy lost is given by

$$E(\omega_\epsilon) - E(\omega_0) = 8(1 - \beta) \tag{6.5}$$

We expect that re-scaling the metrics ω_ϵ around the points p_i we get a Ricci-flat metric on \mathbb{C}^2 with cone angle $2\pi\beta$ along a parabola, as described above. Then 6.5 can be explained by the formation of eight 'bubbles' with energy given by 6.4.

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